# Amenability conditions for certain group algebras

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# Overview

- Brief history of problem
- Introduce example
- Proof in two parts
- Future work

# Preliminaries

**Theorem (Johnson 72)**

*Let $G$ be a LCG. Then $G$ is amenable if and only if $L^1(G)$ is amenable*

- What does it mean for a Banach algebra to be amenable?
- We need to define a few things first.
- Given a Banach algebra $\mathcal{A}$, we can define bi-module actions on $\mathcal{A}\hat{\otimes}\mathcal{A}$.
- Specifically for $a, b, c \in \mathcal{A}$:
    - $a \cdot (b \otimes c) = (ab) \otimes c$
    - $(b \otimes c) \cdot a = b \otimes (ca)$
- We also define the diagonal operator $m : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ by $m(a \otimes b) = ab$.

# Preliminaries

**Definition**

Let $\mathcal{A}$ be a Banach algebra. A **bounded approximate diagonal** (B.A.D.) on $\mathcal{A}$ is a net $(\mu_\gamma)_\gamma \in \mathcal{A}\hat{\otimes}\mathcal{A}$ such that for every $a \in \mathcal{A}$

- $a \cdot \mu_\gamma - \mu_\gamma \cdot \mu \to 0$
- $am(\mu_\gamma) = a$

- We say $\mathcal{A}$ is amenable if it admits a B.A.D.

# Brief History

- Henceforth we assume $G$ is compact.
- We say $G$ is **virtually abelian** if it has an abelian subgroup of finite index.
- We have the following for $A(G)$:

**Theorem (Forrest, Runde 2004)**

*$G$ is virtually abelian if and only if $A(G)$ is amenable.*

- Conjecture first posed by Azimifard-Samei-Spronk in 2009.

**Conjecture**

*$G$ is virtually abelian if and only if $ZL^1(G)$ is amenable.*

- Forward direction was proved by Alaghmandan-Crann in 2017.

# Brief history

- Related conjecture posed by Alaghmandan-Spronk in 2017.
- We define a subalgebra of $A(G)$ called the **central Fourier algebra**:

$$ZA(G) = \{u \in A(G) : u(xyx^{-1}) = u(y), \forall x, y \in G\}$$

**Conjecture**

*$G$ is virtually abelian if and only if $ZA(G)$ is amenable.*

- Forward direction proved in same paper.
- Reverse is known to be true in nice cases:
  - Connected groups
  - (Infinite) product of finite non-abelian groups.

# An example towards an idea

- Consider the $p$-adic group $G = \mathbb{O}_p \rtimes \mathbb{O}_p^* = \begin{pmatrix} \mathbb{O}_p^* & \mathbb{O}_p \\ 0 & 1 \end{pmatrix}$.
- Totally disconnected, non virtually abelian group.
- Consider $ZA(G)$, and its restriction to the subgroup $H = \mathbb{O}_p \rtimes \{1\}$

$$ZA(G)\big|_H = \{u \in A(G) : u(ax, 1) = u(x, 1), \forall x \in \mathbb{O}_p, a \in \mathbb{O}_p^*\}$$

- See a natural isomorphism to the algebra

$$Z_{\mathbb{O}_p^*}A(\mathbb{O}_p) = \{u \in A(\mathbb{O}_p) : u(ax) = u(x), \forall x \in \mathbb{O}_p, a \in \mathbb{O}_p^*\}$$

- This will be our algebra of interest.

# A type of 'diagonal'

- The orbits of $\mathbb{O}_p^* \curvearrowright \mathbb{O}_p$ are $B_n := p^n\mathbb{O}_p \setminus p^{n+1}\mathbb{O}_p$ and $\{0\}$.
- We can construct a "fake diagonal" on these orbits

$$e_n = \sum_{m=0}^{n-1} 1_{B_m} \otimes 1_{B_m} \in Z_{\mathbb{O}_p^*}A(\mathbb{O}_p)\hat{\otimes}Z_{\mathbb{O}_p^*}A(\mathbb{O}_p)$$

- We can also define the f.d. ideal $A_n = \mathrm{span}\{1_{B_m} : 0 \leq m < n\}$, with identity

$$\sum_{m=0}^{n-1} 1_{B_m} = 1_{\mathbb{O}_p} - 1_{p^n\mathbb{O}_p}$$

- Identity has norm at most 2.

# Boundedness along this diagonal

- Suppose $Z_{\mathbb{O}_p^*}A(\mathbb{O}_p)$ is amenable, so it has a B.A.D $(\mu_\gamma)_\gamma$.
- Project $(\mu_\gamma)_\gamma$ onto $A_n \hat{\otimes} A_n$, and let $\eta_n$ be a cluster point.
- We have $\eta_n$ is bounded since identity of $A$ has norm at most 2.
- Can compute coefficients to show that $\eta_n = e_n$.
- So $e_n$ is bounded.

# Unboundedness of this diagonal

- We can also compute $e_n$ explicitly using the Fourier transform.
- The dual group of $\mathbb{O}_p$ is $\mathbb{T}_{p^\infty} = \{x \in S^1 : \exists k \in \mathbb{N}, x^{p^k} = 1\}$.
- Recall we defined $e_n$ as

$$e_n = \sum_{m=0}^{n-1} 1_{B_m} \otimes 1_{B_m} = \sum_{m=0}^{n-1} (1_{p^m \mathbb{O}_p} - 1_{p^{m+1}\mathbb{O}_p}) \otimes (1_{p^m \mathbb{O}_p} - 1_{p^{m+1}\mathbb{O}_p})$$

- Let $H_n = \{x \in \mathbb{T}_{p^\infty} : x^{p^n} = 1\}$.
- The Fourier transform of $e_n$ is

$$d_n = \sum_{m=0}^{n-1} (\frac{1}{p^m} 1_{H_m} - \frac{1}{p^{m+1}} 1_{H_{m+1}}) \otimes (\frac{1}{p^m} 1_{H_m} - \frac{1}{p^{m+1}} 1_{H_{m+1}})$$

- Since $\mathbb{T}_{p^\infty}$ is discrete, it is possible to compute norm directly.

# Unboundedness of this diagonal

- Given $x \in \mathbb{T}_{p^\infty}$, we say it has **order** $k$ if $x^{p^k} = 1$ (and $k$ is minimal).
- With some non-trivial computation, one can show

$$|d_n(x,x)| \geq p^{-2k} - p^{-2n}$$

where $x \in \mathbb{T}_{p^\infty}$ has order $k \leq n$.

- Use this to find lower bound

$$\|d_n\| \geq (1-p^{-1})^2 \left( n - \frac{p^2 - p^{-2n}}{p^2 - 1} \right)$$

- So $d_n$ (and hence $e_n$) is unbounded.

# Summary & future work

- All B.A.D.'s of $Z_{\mathbb{O}_p^*}A(\mathbb{O}_p)$ must 'asymptotically approach' $e_n$.
- But $e_n$ is unbounded, so no B.A.D.'s exist.
- Thus $ZA(\mathbb{O}_p \rtimes \mathbb{O}_p^*)$ is not amenable, affirming the conjecture.
- Hope to use this technique on similar groups.
- In particular with certain kinds of group actions and their orbit structure.

# End

Thank you for listening!