The Spine of Local Fell Groups

Based on joint work with Nico Spronk

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Overview

1. Background
2. Dual Structure on Certain Groups
3. Generalisation to Higher Dimensions

Notation

- As always, G is a locally compact (Hausdorff) group.
- We let $\widehat{G}$ denote the space of irreducible representations (irreps) on G.
- We denote by $A(G)$ and $B(G)$ the Fourier and Fourier-Stieltjes algebras of G.

$$A(G) = \{x \mapsto \langle \lambda(x) f \mid g \rangle : f, g \in L^2(G)\}$$
$$B(G) = \bigcup_{\pi \text{ rep}} \{x \mapsto \langle \pi(x) \xi \mid \eta \rangle : \xi, \zeta \in \mathcal{H}_\pi\}$$

- We let $A_\pi(G) \subseteq B(G)$ be the Fourier space associated to π, where

$$A_\pi(G) := \overline{\text{Span}}\{x \mapsto \langle \pi(x) \xi \mid \zeta \rangle : \xi, \eta \in \mathcal{H}_\pi\}$$

- We let $\mathbb{Q}_p$ and $\mathbb{O}_p$ denote the p-adic numbers and integers, respectively.

History

- $A(G)$ is generally nice and tractable, but $B(G)$ is large and often hard to describe.
- The obvious exception is when G is compact.
- For non-compact G, when is $B(G)$ "easy" to describe?

Theorem (Runde-Spronk '04)

Let G be a locally compact group (LCG). Then $AM_{op}(B(G)) < 5$ iff G is compact.

- Originally conjectured to be true if 5 is replaced with ∞.

Fell Group

Theorem (Runde-Spronk '07)

Let $G_p = \mathbb{Q}_p \rtimes \mathbb{O}_p^*$. *Then* $AM_{op}(B(G_p)) = 5$.

- $G_p = \mathbb{Q}_p \rtimes \mathbb{O}_p^*$ is often called the *Fell group*.
- Let us examine the structure of $B(G_p)$.

Theorem (Baggett '72, Runde-Spronk '07)

Let $G_p = \mathbb{Q}_p \rtimes \mathbb{O}_p^*$. *Then* $B(G_p) = A(G_p) \oplus A(\mathbb{O}_p^*) \circ q$ *where* $q : G \to \mathbb{O}_p^*$ *is the quotient map.*

- Crucially, $B(G_p)$ is constructed as a sum of Fourier algebras (though of other groups).

The Spine of a Group

- The structure of $B(G_p)$ leads us to consider the following algebra.

Definition (Ilie-Spronk '06)

Let G be a LCG. Then we define the **spine** of $B(G)$ to be

$$A^*(G) := \overline{\mathrm{Span}}\{A(H) \circ \eta : H \text{ is a LCG and } \eta : G \to H \text{ a homomorphism}\}$$

- Thus for G_p, $A^*(G_p) = B(G_p) = A(G_p) \oplus A(\mathbb{O}_p^*)$

The Spine of a Group

- We say a non-compact group G is **spinal** if $B(G) = A^*(G)$.
- Another notion of "smallness", indicates that $B(G)$ is perhaps more tractable.

Question

When is a non-compact group spinal?

- We know that G_p is spinal, and so are equivalent versions on totally disconnected local fields.
- Outside of these, no other examples are known.
- We look to the structure of $B(G_p)$ to understand this property.

Mackey Machine

- The structure of $B(G_p)$ was determined using the aid of the so-called *Mackey machine*.
- When $G = A \rtimes H$, where A is abelian and the action $H \curvearrowright \widehat{A}$ is 'nice' (regular), then the Mackey machine describes the irreps $\widehat{G}$.
- In particular, the irreps are determined by induced reps.
- The exact construction depends on the orbit and stabiliser structure of the dual action $H \curvearrowright \widehat{A}$.
    - For $h \in H$ and $\varphi \in \widehat{A}$, we set $k \cdot \varphi(a) := \varphi(k^{-1} \cdot a)$ for $a \in A$.
- The full description is quite technical and gets a little messy.
- However, it is easier to work with for certain types of group actions.

Cheap Groups

Definition

Let A be abelian and K a compact group, both second countable.We say $G = A \rtimes K$ is **cheap** if for the action $K \curvearrowright \widehat{A}$, the stabilisers of non-trivial elements are always trivial.

- Equivalently, G is trivial if whenever $k \cdot \varphi = \varphi$, then either k or φ are trivial.
- We can apply the Mackey machine to cheap groups.

Theorem (Spronk-V. '25)

Let $G = A \rtimes K$ be a cheap group and $q : G \to K$ the canonical quotient map. Then any $\pi \in \widehat{G}$ comes in one of two forms:

- *$\pi = \rho \circ q$ for some $\rho \in \widehat{K}$, in which case π is always finite dimensional.*
- *$\pi = \mathrm{Ind}_A^G(\varphi)$ for some non-trivial $\varphi \in \widehat{A}$, in which case π is infinite dimensional iff K is infinite. Moreover, $\mathrm{Ind}_A^G(\varphi) \sim \mathrm{Ind}_A^G(\psi)$ iff $\varphi \in K \cdot \psi$.*

Cheap Group Deals

- This gives a nice (topological) structure on the dual space $\widehat{G}$:

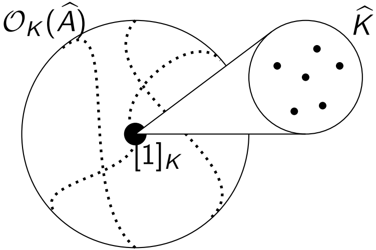

Figure: The dual space of a cheap group.

- This construction is akin to the 'line with two origins'.
- It is not Hausdorff, but it is T_1.
- In particular, this means G is *Type I*.

Infinite Cheapness

- When K is infinite, we say G is **infinitely cheap**.
- In this case, the irreps split nicely into two categories:
  - Finite irreps from K of the form $\rho \circ q$.
  - Infinite irreps from A of the form $\mathrm{Ind}_A^G(\varphi)$.
- Let $B_\infty(G)$ be the ideal of $B(G)$ given by the coefficient functions of purely infinite reps ($=$ no finite subreps).

Proposition (Spronk-V. '25)

Let G be infinitely cheap. Then $G^{ap} = K$.

Corollary

Let G be infinitely cheap. Then $B(G) = B_\infty(G) \oplus A(K) \circ q$. Moreover, $B(G)$ is generated by elements of the form $x \mapsto \langle \mathrm{Ind}_A^G(\sigma)(x)\xi \mid \eta \rangle$ where σ is (any) rep of A.

Local Fell Groups

Theorem (Fell group case: Baggett '72, Walter '77. General case: Spronk-V. '25)

Let $G = A \rtimes K$ be infinitely cheap. If every non-zero orbit is open, and the orbit space is countable, then $B(G) = A(G) \oplus A(K) \circ q$. In particular G is spinal.

- The action of $\mathbb{O}_p^* \curvearrowright \mathbb{Q}_p$ has open orbits of the form $p^n\mathbb{O}_p \setminus p^{n+1}\mathbb{O}_p$. In particular, the (non-zero) orbit space is homeomorphic to $\mathbb{Z}$.
- Since G_p is infinitely cheap, then G_p is spinal.
- This argument also works for $G = \mathbb{K} \rtimes \mathcal{U}$, where $\mathbb{K}$ is a totally disconnected local field and $\mathcal{U} = \{x \in \mathbb{K} : |x| = 1\}$.
- We call such groups *local Fell groups*.

Question

Are there groups $G = A \rtimes K$ other than local Fell groups for which the conditions of the above theorem hold?

Higher Dimensions

- Many different ways to generalise, one may try higher dimensions.
- We will look at $\mathcal{G} = \mathbb{K}^2 \rtimes \mathcal{U}$.
- This group $\mathcal{G}$ has no open orbits. Instead, the non-zero orbit space is of the form $\mathbb{K}^\circ \times \mathbb{Z}$ where $\mathbb{K}^\circ$ is the one-point compactification of $\mathbb{K}$.
- However, $\mathcal{G}$ is still infinitely cheap, so we still have $B(\mathcal{G}) = B_\infty(\mathcal{G}) \oplus A(\mathcal{U}) \circ q$.
- So our goal is to understand $B_\infty(\mathcal{G})$.

Kernel Lines

- Infinite irreps of $\mathcal{G}$ are of the form $\pi_\varphi = \mathrm{Ind}_{\mathbb{K}^2}^{\mathcal{G}}(\varphi)$ where $\varphi \in \widehat{\mathbb{K}^2}$.
- Given such a π_φ, there is a line $L \subseteq \mathbb{K}^2$ such that $L \subseteq \ker \pi_\varphi$.
- It follows that if $q_L : \mathcal{G} \to \mathcal{G}/L$ is the quotient map, then

$$A_{\pi_\varphi}(\mathcal{G}) \subseteq A(\mathcal{G}/L) \circ q_L$$

- In particular, if π is a direct sum of irreps, then

$$A_\pi(\mathcal{G}) \subseteq \bigoplus_{L \text{ line}} A(\mathcal{G}/L) \circ q_L$$

- What if π is not a direct sum?

Direct Integrals

- For a general rep π, we can write it as a *direct integral*:

$$\pi = \int_{\widehat{\mathbb{K}^2}}^{\oplus} \pi_\varphi \, \mathrm{d}\mu(\varphi)$$

  for some positive measure μ. We write $\pi_\mu = \pi$.
- This is the usual direct sum when μ is discrete.
- We say μ is **continuous** if it has no atoms.
- If $u \in A_{\pi_\mu}(\mathcal{G})$ and $v \in A_{\pi_\nu}(\mathcal{G})$ where μ is continuous, then $uv \in A(\mathcal{G})$.
- Setting $\nu = \mu$ and $v = u$, this shows that $u \in C_0(\mathcal{G})$.

Putting it All Together

- Define $B_0(G) := B(G) \cap C_0(G)$. This is called the **Rajchman algebra** of G.

Theorem (Spronk-V. '25)

Let $\mathcal{G} = \mathbb{K}^2 \rtimes \mathcal{U}$. Then

$$B(\mathcal{G}) = B_0(\mathcal{G}) \oplus \left[\bigoplus_{L \text{ line}} A(\mathcal{G}/L) \circ q_L \right] \oplus A(\mathcal{U}) \circ q$$

Moreover if $u, v \in B_0(\mathcal{G})$, then $uv \in A(\mathcal{G})$. In other words, $A(\mathcal{G})$ is the radicaliser of $B_0(\mathcal{G})$ in $B(\mathcal{G})$.

One Final Note

- One can show that in general, $B_0(G) \cap A^*(G) = A(G)$, and so

$$A^*(\mathcal{G}) = A(\mathcal{G}) \oplus \left[\bigoplus_{L \text{ line}} A(\mathcal{G}/L) \circ q_L \right] \oplus A(\mathcal{U}) \circ q$$

- One can also show that $B_0(\mathcal{G})$ is strictly larger than $A(\mathcal{G})$, and so $\mathcal{G}$ is *not* spinal.
- However, $B(\mathcal{G})$ is close to being $A^*(\mathcal{G})$, with many products landing inside $A(\mathcal{G})$.

Question

How does $B(\mathcal{G})$ behave for $\mathcal{G} = \mathbb{K}^n \rtimes \mathcal{U}$ for $n > 2$?

Thank you