Sample Problems

Q1. Prove that $\text{Span}\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$ in $\mathbb{R}^3$.

(Remember that to prove two sets are equal, you must show that they are subsets of each other.)

Q2. Let $(1,1,0)$ and $(3,-2,1)$ be two points on a line $\mathcal{L}$ in $\mathbb{R}^3$.

(a) Find a vector equation for $\mathcal{L}$.

(b) Find parametric equations for $\mathcal{L}$.

(c) Determine whether the point $(-1,4,-1)$ is on $\mathcal{L}$.

(d) Determine whether the point $(2,0,2)$ is on $\mathcal{L}$.

Q3. Let $\mathcal{L}$ be the line $\mathbb{R}^2$ with the following equation: $\vec{\ell} = \vec{u} + t\vec{v}, t \in \mathbb{R}$, where

$$\vec{u} = \begin{bmatrix}0\\5\end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix}-2\\4\end{bmatrix}.$$

(a) Show that the vector $\vec{x} = [4 \ -3]^T$ lies on $\mathcal{L}$.

(b) Find a **unit** vector $\vec{n}$ which is orthogonal to $\vec{v}$.

(c) Compute $\vec{y} = \text{proj}_{\vec{n}}(\vec{x})$ and show that this vector lies on $\mathcal{L}$.

(d) Prove that $\vec{y}$ is the vector on $\mathcal{L}$ with the smallest norm. That is, prove that $\|\vec{y}\| \leq \|\vec{\ell}\|$ for all $\vec{\ell} \in \mathcal{L}$. (Hint: show that you can write the vectors of $\mathcal{L}$ in the form $\vec{\ell} = \vec{y} + s\vec{v}, s \in \mathbb{R}$ and use Q2a from Assignment 1).

Q4. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lines in $\mathbb{R}^3$ with the respective vector equations:

$$\vec{\ell}_1 = \begin{bmatrix}2\\3\\2\end{bmatrix} + t\begin{bmatrix}1\\2\\-1\end{bmatrix}, t \in \mathbb{R} \qquad \text{and} \qquad \vec{\ell}_2 = \begin{bmatrix}2\\1\\3\end{bmatrix} + t\begin{bmatrix}2\\2\\-1\end{bmatrix}, t \in \mathbb{R}.$$

(a) Find a point A which lies on the intersection of $\mathcal{L}_1$ and $\mathcal{L}_2$.

(b) Let $\mathcal{P}$ be a plane which contains both $\mathcal{L}_1$ and $\mathcal{L}_2$. Find a vector equation for $\mathcal{P}$.

(c) Find a scalar equation for $\mathcal{P}$.

Q5. (a) Determine all real values a and b such that $\begin{bmatrix} 1 \\ 3a \end{bmatrix} \notin \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} b \\ 4 \end{bmatrix} \right\}$ in $\mathbb{R}^2$.

(b) Determine the solution set, S, to the following system of linear equations.

$$\begin{array}{rrrrcl} 2x_1 & -x_2 & +2x_3 & +4x_4 & = & 0 \\ 2x_1 & -x_2 & & +3x_4 & = & 0 \end{array}.$$

Express S as the span of one or more vectors.

Q6. Let $A \in M_3(\mathbb{R})$ be the following matrix:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}.$$

(a) Compute the inverse A^{-1}.

(b) For a fixed $\vec{b} \in \mathbb{R}^3$, how many solutions does the system $A\vec{x} = \vec{b}$ have?

(c) Solve the system $A\vec{x} = \vec{c}$ where $\vec{c} = [1\ 1\ 1]^T$. Do not use the Gauss-Jordan algorithm.

Q7. Let $A \in M_n(\mathbb{F})$, and suppose that $A^3 + 2A^2 - A + I_n = \mathcal{O}_n$. Prove that A is invertible. What is its inverse?

Q8. For each of the following transformations, either prove that it is linear, or find a counterexample to show it is not.

(a) The transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ xy \end{bmatrix}$.

(b) The transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x+2y+z \\ x+z \end{bmatrix}$.

(c) The transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x-y \\ -x \\ 2+3y \end{bmatrix}$.

(d) For a fixed $\vec{z} \in \mathbb{R}^3$, the transformation $T_{\vec{z}} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T_{\vec{z}}(\vec{x}) = \vec{z} \times \vec{x}$.

Q9. (a) Determine the kernel of the linear transformation $T_1\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ x_2 + 2x_3 \\ x_3 \end{bmatrix}.$$

(b) Is T_1 one-to-one? Justify your answer.

(c) Determine the range of the linear transformation $T_2\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 + x_3 \end{bmatrix}.$$

(d) Is T_2 onto? Justify your answer.

Q10. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose that there exists a map $S : \mathbb{R}^m \to \mathbb{R}^n$ with the property that that

$$S(\vec{x}) \cdot \vec{v} = \vec{x} \cdot T(\vec{v})$$

for all $\vec{x} \in \mathbb{R}^m$ and $\vec{v} \in \mathbb{R}^n$.

(a) Prove that S is a linear transformation.

(b) Since both T and S are linear, we can find matrices A and B such that $T = T_A$ and $S = T_B$. Prove that $B = A^T$.

Q11. Determine, with justification, whether the following sets are subspaces of $\mathbb{R}^4$.

(a) $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_1 + x_2 - x_3 - x_4 = 0 \right\}$.

(b) $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : |x_1| + |x_2| - |x_3| - |x_4| = 0 \right\}$.

(c) $S = \left\{ \begin{bmatrix} x \\ x^2 \\ y \\ y^2 \end{bmatrix} \in \mathbb{R}^4 : x, y \in \mathbb{R} \right\}$.

Q12. Let V and W be subspaces of $\mathbb{R}^n$. Prove or disprove the following.

(a) $V \cap W$ is a subspace of $\mathbb{R}^n$.

(b) $V \cup W$ is a subspace of $\mathbb{R}^n$.

(c) V^c (the complement of V in $\mathbb{R}^n$) is a subspace of $\mathbb{R}^n$.

Q13. Determine, with justification, whether the following sets are linearly independent.

(a) $S_1 = \left\{ \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} -6 \\ -9 \\ 3 \end{bmatrix} \right\}$.

(b) $S_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$.

(c) $S_3 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$.

Q14. Using Proposition 8.3.2 from the notes, we can rewrite Question 2b from Written Assignment 7 as the following lemma (do **not** prove this):

Lemma. *Let $A \in M_n(\mathbb{F})$. If $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of A with distinct eigenvalues λ_1 and λ_2, then the set $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.*

Now let $A \in M_n(\mathbb{F})$, and let $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ be eigenvectors of A with *pairwise distinct* eigenvalues λ_1, λ_2 and λ_3 (in other words, $\lambda_i \neq \lambda_j$ for all $i \neq j$). Prove that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent. You may use Lemma 1 in your proof.

Q15. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Prove that the following are equivalent:

(i) T is injective.

(ii) For all vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$, if the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \subseteq \mathbb{R}^n$ is linearly independent, then the corresponding set $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\} \subseteq \mathbb{R}^m$ is also linearly independent.

Q16. Determine whether the matrix

$$A = \begin{bmatrix} 3 & 2 & -2 & -2 \\ -1 & 2 & 2 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

is diagonalizable over $\mathbb{R}$ and/or $\mathbb{C}$.

Q17. Let $A \in M_4(\mathbb{C})$ and suppose it is diagonalizable over $\mathbb{C}$. Furthermore suppose it has eigenvalues $\lambda = 1, 2, 3$, and that these are all its eigenvalues. How many diagonal matrices D are there which are similar to A?

Q18. Consider $P_n(\mathbb{F})$, the vector space of polynomials over $\mathbb{F}$ of degree at most n. Define the function $D : P_n(\mathbb{F}) \to P_n(\mathbb{F})$ as follows: for a polynomial $p(x) = \sum_{i=0}^{n} a_i x^i$ where $a_i \in \mathbb{F}$, set

$$Dp(x) = \sum_{i=1}^{n} i a_i x^{i-1}$$

(a) Show that D is a linear map.

(b) What is the range of D? Is D surjective?

(c) What is the kernel of D? Is D injective?

Q19. Consider the plane $\mathcal{P}$ in $\mathbb{R}^3$ with scalar equation $3x - y + 2z = 0$. For each of the following lines, find all intersection points with $\mathcal{P}$, and describe what this set looks like geometrically (eg. "a line through $\vec{u}$ with direction vector $\vec{v}$").

(a) $\mathcal{L}_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

(b) $\mathcal{L}_2 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} : t \in \mathbb{R} \right\}$

Q20. Let $\mathcal{P}$ be a plane in $\mathbb{R}^n$ through the origin with direction vectors $\vec{u}$ and $\vec{v}$. Assume moreover that $\vec{u}$ and $\vec{v}$ are orthogonal and are unit vectors. We will define the projection of a vector $\vec{z}$ onto $\mathcal{P}$ by

$$\text{proj}_{\mathcal{P}}(\vec{z}) = \text{proj}_{\vec{u}}(\vec{z}) + \text{proj}_{\vec{v}}(\vec{z})$$

(a) Prove that $\text{proj}_{\mathcal{P}}$ is linear. That is, prove that $\text{proj}_{\mathcal{P}}(a\vec{x}+b\vec{y}) = a\,\text{proj}_{\mathcal{P}}(\vec{x})+b\,\text{proj}_{\mathcal{P}}(\vec{y})$ for all $a, b \in \mathbb{R}$ and $\vec{x}, \vec{y} \in \mathbb{R}^n$.

(b) Show that if $\vec{w} \in \mathcal{P}$, then $\text{proj}_{\mathcal{P}}(\vec{w}) = \vec{w}$.

(c) Use the previous result to show that if $\vec{x} \in \mathbb{R}^n$ and $\vec{w} \in \mathcal{P}$, then $\text{proj}_{\mathcal{P}}(\vec{x}) \cdot \vec{w} = \vec{x} \cdot \vec{w}$.

Q21. Let

$$\vec{x} = \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & -2 \\ -1 & -3 \end{bmatrix}, C = \begin{bmatrix} 0 & -2 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Compute the following expressions if they exist. For each expression, if it is not valid write 'NO', otherwise if it is valid write 'YES' **and** compute it.

(i) $\vec{x} - 3\vec{y}$	(ii) $\lVert\vec{x}\rVert$	(iii) $\vec{x} \cdot \vec{y} \cdot \vec{y}$	(iv) $\text{proj}_{\vec{x}}(\vec{y})$
(v) $B\vec{x}$	(vi) $C\vec{x}$	(vii) $A + 2B$	(viii) $2A - C^T$
(ix) AB	(x) BA	(xi) $\det(B)$	(xii) $\det(C)$

Q22. Let $A \in M_3(\mathbb{R})$ be the following matrix.

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 7 & -5 \\ -2 & -3 & 2 \end{bmatrix}.$$

(a) Find A^{-1}. Show your working, including labelling row operations, if any.

(b) Consider the following system of equations

$$\begin{aligned} x_1 + 3x_2 - 2x_3 &= a \\ 2x_1 + 7x_2 - 5x_3 &= b \\ -2x_1 - 3x_2 + 2x_3 &= c \end{aligned}$$

for some $a, b, c \in \mathbb{R}$. Why is this system always consistent?

(c) Give a solution to this system in terms of a, b, and c.

(d) Find an expression for a matrix B with eigenpairs $\left(-1, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}\right)$, $\left(1, \begin{bmatrix} 3 \\ 7 \\ -3 \end{bmatrix}\right)$, and $\left(1, \begin{bmatrix} -2 \\ -5 \\ 2 \end{bmatrix}\right)$. You do not have to simplify your answer.

Q23. Let $\mathcal{P}$ be a plane in $\mathbb{R}^3$ through the origin with normal vector $\vec{n}$. Suppose that $\mathcal{L}$ is a line (also in $\mathbb{R}^3$) going through $\vec{u}$ with direction vector $\vec{v}$. Suppose further that $\vec{n} \cdot \vec{v} = 0$.

(a) Write down a vector equation for the line $\mathcal{L}$.

(b) Prove that if $\vec{u} \in \mathcal{P}$ then $\mathcal{L}$ is contained in $\mathcal{P}$.

Q24. Recall that the **trace** of a matrix is the sum of the diagonal entries. Let $A, B \in M_n(\mathbb{F})$ with entries

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}.$$

(a) Express $\mathrm{tr}(A)$ in terms of the entries of A.

(b) Express the (i, j)-th entry of matrix product AB, in terms of the entries of A and B.

(c) Use the previous two results to find an expression for $\mathrm{tr}(AB)$ in terms of the entries of A and B.

(d) Use part (c) to show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(e) Use part (d) to show that if $S, T \in M_n(\mathbb{F})$ are similar matrices, then $\mathrm{tr}(S) = \mathrm{tr}(T)$.