Solutions

Q1. Prove that $\text{Span}\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$ in $\mathbb{R}^3$.

(Remember that to prove two sets are equal, you must show that they are subsets of each other.)

Solution: Firstly, let us notice that, by inspection, we have that $[3\ 1\ 0]^T$ is a linear combination of $[1\ 1\ 2]^T$ and $[1\ 0\ -1]^T$. Or, more explicitly,

$$\begin{bmatrix}3\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix} + 2\begin{bmatrix}1\\0\\-1\end{bmatrix} \tag{1}$$

We shall keep this in mind for later.

Let us start by showing that $\text{Span}\{[1\ 1\ 2]^T, [1\ 0\ -1]^T\} \subseteq \text{Span}\{[3\ 1\ 0]^T, [1\ 0\ -1]^T\}$. To this end, let us take some $\vec{u} \in \text{Span}\{[1\ 1\ 2]^T, [1\ 0\ -1]^T\}$, which means we can write

$$\vec{u} = a\begin{bmatrix}1\\1\\2\end{bmatrix} + b\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

for some $a, b \in \mathbb{R}$. From the equality in (1), let us define a new variable $c = b - 2a$. We then have that

$$\begin{aligned}\vec{u} &= a\begin{bmatrix}1\\1\\2\end{bmatrix} + (c+2a)\begin{bmatrix}1\\0\\-1\end{bmatrix}\\ &= a\left(\begin{bmatrix}1\\1\\2\end{bmatrix} + 2\begin{bmatrix}1\\0\\-1\end{bmatrix}\right) + c\begin{bmatrix}1\\0\\-1\end{bmatrix}\\ &= a\begin{bmatrix}3\\1\\0\end{bmatrix} + c\begin{bmatrix}1\\0\\-1\end{bmatrix}\end{aligned}$$

From this it follows that $\vec{u}$ is a linear combination of $[3\ 1\ 0]^T$ and $[1\ 0\ -1]^T$. Hence $\vec{u} \in \text{Span}\{[3\ 1\ 0]^T, [1\ 0\ -1]^T\}$ and so we have proved that $\text{Span}\{[1\ 1\ 2]^T, [1\ 0\ -1]^T\} \subseteq \text{Span}\{[3\ 1\ 0]^T, [1\ 0\ -1]^T\}$.

The other direction is similar, we start with some $\vec{v} \in \text{Span}\{[3\ 1\ 0]^T, [1\ 0\ -1]^T\}$ and write

$$\vec{v} = a\begin{bmatrix}3\\1\\0\end{bmatrix} + b\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

Again keeping (1) in mind, we define $c = b + 2a$, so that

$$\begin{aligned}\vec{v} &= a \begin{bmatrix}3\\1\\0\end{bmatrix} + (c-2a)\begin{bmatrix}1\\0\\-1\end{bmatrix}\\ &= a\left(\begin{bmatrix}3\\1\\0\end{bmatrix} - 2\begin{bmatrix}1\\0\\-1\end{bmatrix}\right) + c\begin{bmatrix}1\\0\\-1\end{bmatrix}\\ &= a\begin{bmatrix}1\\1\\2\end{bmatrix} + c\begin{bmatrix}1\\0\\-1\end{bmatrix}\end{aligned}$$

Thus $\vec{v}$ is a linear combination of $[1\ 1\ 2]^T$ and $[1\ 0\ {-1}]^T$ and so $\vec{v} \in \text{Span}\{[3\ 1\ 0]^T, [1\ 0\ \ -1]^T\}$. This shows the other inclusion, and so we conclude that $\text{Span}\{[1\ 1\ 2]^T, [1\ 0\ {-1}]^T\} = \text{Span}\{[3\ 1\ 0]^T, [1\ 0\ \ -1]^T\}$

Q2. Let $(1, 1, 0)$ and $(3, -2, 1)$ be two points on a line $\mathcal{L}$ in $\mathbb{R}^3$.

(a) Find a vector equation for $\mathcal{L}$.

(b) Find parametric equations for $\mathcal{L}$.

(c) Determine whether the point $(-1, 4, -1)$ is on $\mathcal{L}$.

(d) Determine whether the point $(2, 0, 2)$ is on $\mathcal{L}$.

Solution: (a) First we find a direction vector $\vec{v}$ for the line. For instance, this can be the the vector which starts at $(1, 1, 0)$ and finishes at $(3, -2, 1)$. In this case that is

$$\vec{v} = \begin{bmatrix}3\\-2\\1\end{bmatrix} - \begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}2\\-3\\1\end{bmatrix}$$

We can also choose the initial vector to be $\vec{u} = [1\ 1\ 0]^T$. With this, we obtain the following vector equation for $\mathcal{L}$:

$$\vec{\ell} = \vec{u} + t\vec{v} = \begin{bmatrix}1\\1\\0\end{bmatrix} + t\begin{bmatrix}2\\-3\\1\end{bmatrix}, t \in \mathbb{R}$$

(b) To get the parametric equation, we simply extract the component-wise equations from the vector equation. In this case these are

$$\begin{aligned}\ell_1 &= 1 + 2t\\ \ell_2 &= 1 - 3t, \quad t \in \mathbb{R}\\ \ell_3 &= t\end{aligned}$$

(c) To determine if $(-1, 4, -1)$ is on this line, we would need to check that it satisfies the parametric equations above. From the third equation, we see that we must have $t = -1$. Checking the other equations we see that $1 + 2t = -1$ and $1 - 3t = 4$, so this point satisfies the parametric equations. Hence it is on the line $\mathcal{L}$.

(d) To determine if $(2, 0, 2)$ is on this line, we would need to check that it satisfies the parametric equations above. From the third equation, we see that we must have $t = 2$. However, checking the first equation we see that $1 + 2t = 5 \neq 2$, so this point does not satisfy the parametric equations. Hence it is not on the line $\mathcal{L}$.

Q3. Let $\mathcal{L}$ be the line $\mathbb{R}^2$ with the following equation: $\vec{\ell} = \vec{u} + t\vec{v}, t \in \mathbb{R}$, where

$$\vec{u} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

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(a) Show that the vector $\vec{x} = [4 \ \ -3]^T$ lies on $\mathcal{L}$.

(b) Find a **unit** vector $\vec{n}$ which is orthogonal to $\vec{v}$.

(c) Compute $\vec{y} = \text{proj}_{\vec{n}}(\vec{x})$ and show that this vector lies on $\mathcal{L}$.

(d) Prove that $\vec{y}$ is the vector on $\mathcal{L}$ with the smallest norm. That is, prove that $\|\vec{y}\| \leq \|\vec{\ell}\|$ for all $\vec{\ell} \in \mathcal{L}$. (Hint: show that you can write the vectors of $\mathcal{L}$ in the form $\vec{\ell} = \vec{y} + s\vec{v}, s \in \mathbb{R}$ and use Q2a from Assignment 1).

Solution: (a) If we take $t = -2$, we see that

$$\vec{u} + t\vec{v} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \vec{x}$$

and so $\vec{x}$ lies on $\mathcal{L}$.

(b) Let $\vec{n} = [n_1 \ n_2]^T$. Since $\vec{n}$ is orthogonal to $\vec{v}$, we know that $\vec{n} \cdot \vec{v} = 0$, and so $-2n_1 + 4n_2 = 0$. Hence we get $n_1 = 2n_2$, and so $\vec{n} = [2n_2 \ n_2]^T$.
Now, since $\vec{n}$ is a unit vector, we must have $\|v_n\| = 1$, and so $1 = \sqrt{4n_2^2 + n_2^2} = \sqrt{5n_2^2}$. This gives us that $n_2 = 1/\sqrt{5}$, and so

$$\vec{n} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(Note we could have chosen $n_2 = -1/\sqrt{5}$ and this would have given us a valid answer as well).

(c) We compute

$$\vec{y} = \frac{\vec{x} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \left(\begin{bmatrix} 4 \\ -3 \end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{5}\left(\begin{bmatrix} 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{5}(8-3)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Furthermore we need to verify $\vec{y}$ is on the line $\mathcal{L}$. If we set $t = -1$, we see that

$$\vec{u} + t\vec{v} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \vec{y}$$

and so vy lies on $\mathcal{L}$.

(d) Consider the vector equation for $\mathcal{L}$, $\vec{\ell} = \vec{u} + t\vec{v}, t \in \mathbb{R}$. If we define $s = t + 1$, then we have

$$\vec{\ell} = \vec{u} + t\vec{v} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + (s-1)\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + s\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \vec{y} + s\vec{v}$$

This gives us an alternative vector equation of the line $\mathcal{L}$.
Now, notice that $\vec{y} \cdot \vec{v}$ = -4 + 4 = 0, so these vectors are orthogonal. We can then use the result of Q2a in Assignment 1 to see that

$$\|\vec{\ell}\|^2 = \|\vec{y} + s\vec{v}\|^2 = \|\vec{y}\|^2 + \|s\vec{v}\|^2 \geq \|\vec{y}\|^2 \qquad (\text{as } \|s\vec{v}\|^2 \geq 0)$$

From this it follows that $\|\vec{y}\| \leq \|\vec{\ell}\|$, and so it is the vector on $\mathcal{L}$ with minimal norm.

Q4. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lines in $\mathbb{R}^3$ with the respective vector equations:

$$\vec{\ell}_1 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, t \in \mathbb{R} \qquad \text{and} \qquad \vec{\ell}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, t \in \mathbb{R}.$$

(a) Find a point A which lies on the intersection of $\mathcal{L}_1$ and $\mathcal{L}_2$.

(b) Let $\mathcal{P}$ be a plane which contains both $\mathcal{L}_1$ and $\mathcal{L}_2$. Find a vector equation for $\mathcal{P}$.

(c) Find a scalar equation for $\mathcal{P}$.

Solution: (a) We need to find a point which satisfies the vector equations for both $\mathcal{L}_1$ and $\mathcal{L}_2$. In other words, we want $\vec{\ell}_1 = \vec{\ell}_2$; writing this out as parametric equations we get

$$\begin{aligned} 2 + t &= 2 + 2s \\ 3 + 2t &= 1 + 2s \\ 2 - t &= 3 - s \end{aligned}$$

After some rearrangement we get

$$\begin{aligned} t - 2s &= 0 \\ 2t - 2s &= -2 \\ -t + s &= 1 \end{aligned}$$

The first equation gives us that $t = 2s$. Substituting this in the second equation, we have that $-2 = 2t - 2s = 2s$, and so we get $s = -1$ and $t = -2$. A quick sanity check also shows that these values satisfy the third equation.
To actually compute A, we simply substitute one of these values in. If we let $\vec{a}$ be the vector corresponding to the point A, then

$$\vec{a} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

So the point A is $(0, -1, 4)$.

(b) To find the vector equation for this plane, we need a point on the plane, as well as two non-zero direction vectors (which are not scalar multiples of each other). We already have the point A on the plane, and we have the two direction vectors of the lines $\mathcal{L}_1$ and $\mathcal{L}_2$. Using these we get the following vector equation of the plane

$$\vec{p} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

(c) To compute the scalar equation, we first need to find the normal vector. This is given by

$$\vec{n} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}$$

Now we know the normal form of the plane is given by

$$\vec{n} \cdot (\vec{p} - \vec{a}) = 0$$

Expanding this computation we get

$$\begin{aligned} \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} \right) &= 0 \\ \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y+1 \\ z-4 \end{bmatrix} &= 0 \\ -(y+1) - 2(z-4) &= 0 \\ -y - 2z &= -7 \\ y + 2z &= 7 \end{aligned}$$

This is the scalar equation of the plane $\mathcal{P}$.

Q5. (a) Determine all real values a and b such that $\begin{bmatrix} 1 \\ 3a \end{bmatrix} \notin \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} b \\ 4 \end{bmatrix} \right\}$ in $\mathbb{R}^2$.

(b) Determine the solution set, S, to the following system of linear equations.

$$\begin{array}{rrrrcl} 2x_1 & -x_2 & +2x_3 & +4x_4 & = & 0 \\ 2x_1 & -x_2 & & +3x_4 & = & 0 \end{array}.$$

Express S as the span of one or more vectors.

Solution: (a) We know that the vector $[1\ 3a]^T$ is in the span of $[2\ 2]^T$ and $[b\ 4]^T$ if and only if it is a linear combination of these vectors, That is, if and only if there is $s, t \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 \\ 3a \end{bmatrix} = s \begin{bmatrix} 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} b \\ 4 \end{bmatrix}$$

Writing this as a system of linear equations we get

$$2s + bt = 1$$
$$2s + 4t = 3a$$

That is, the vector $[1\ 3a]^T$ is in the span of $[2\ 2]^T$ and $[b\ 4]^T$ if and only if the above system has a solution.
First, notice that if $b = 4$, then the left hand side will be the same for both equations. This will fail to have a solution if and only if $1 \neq 3a$.
On the other hand, if we assume $b \neq 4$, then we can subtract the second equation from the first, and obtain $(b-4)t = 1 - 3a$. Since $b \neq 4$, then we get $t = \frac{1-3a}{b-4}$. Since $2s + 4t = 3a$, then we get that

$$s = \frac{3a - 4t}{2} = \frac{3a}{2} - 2\frac{1-3a}{b-4} = \frac{3a(b-4) - 4(1-3a)}{2(b-4)} = \frac{3ab - 4}{2b - 8}$$

Plugging back into our original system verifies that this choice of s and t is a solution. In other words, when $b \neq 4$, we always obtain a solution.
Thus $\begin{bmatrix} 1 \\ 3a \end{bmatrix} \notin \text{Span}\left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} b \\ 4 \end{bmatrix} \right\}$ in $\mathbb{R}^2$, precisely when the above system has no solution, which is true precisely when $b = 4$ and $a \neq 1/3$.

(b) Let $[x_1\ x_2\ x_3\ x_4]^T$ be a solution to this system. If we subtract the second equation from the last, we obtain $2x_3 + x_4 = 0$, or simplifying we get $x_4 = -2x_3$. Substituting this into (either) equation we get

$$2x_1 - x_2 - 6x_3 = 0$$

We can freely assign two of these variables, after which the third will be fixed. So let $s = x_2$ and $t = x_3$, after which we get $x_1 = s/2 + 3t$. Furthermore we have $x_4 = -2x_3 = -2t$. So we can write the solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}s + 3t \\ s \\ t \\ -2t \end{bmatrix} = s \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

In other words, the solution is a linear combination of $\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \\ -2 \end{bmatrix}$, so the solution set is

$$S = \text{Span}\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

Q6. Let $A \in M_3(\mathbb{R})$ be the following matrix:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

(a) Compute the inverse A^{-1}.
(b) For a fixed $\vec{b} \in \mathbb{R}^3$, how many solutions does the system $A\vec{x} = \vec{b}$ have?
(c) Solve the system $A\vec{x} = \vec{c}$ where $\vec{c} = [1\ 1\ 1]^T$. Do not use the Gauss-Jordan algorithm.

Solution: (a) We begin by reducing the super-augmented matrix $[A|I_3]$.

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 1 & 0 & 0 \end{array}\right] \qquad R_1 \leftrightarrow R_3$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & -5 & -2 & 0 & 1 & -2 \\ 0 & -4 & -2 & 1 & 0 & -3 \end{array}\right] \qquad \begin{array}{c} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2/5 & 0 & -1/5 & 2/5 \\ 0 & -4 & -2 & 1 & 0 & -3 \end{array}\right] \qquad R_2 \to \frac{-1}{5}R_2$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2/5 & 0 & -1/5 & 2/5 \\ 0 & 0 & -2/5 & 1 & -4/5 & -7/5 \end{array}\right] \qquad R_3 \to R_3 + 4R_2$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2/5 & 0 & -1/5 & 2/5 \\ 0 & 0 & 1 & -5/2 & 2 & 7/2 \end{array}\right] \qquad R_3 \to \frac{-5}{2}R_3$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 5/2 & -2 & -5/2 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -5/2 & 2 & 7/2 \end{array}\right] \qquad \begin{array}{c} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - \frac{2}{5}R_3 \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -5/2 & 2 & 7/2 \end{array}\right] \qquad R_1 \to R_1 - 2R_2$$

This gives us that the inverse of A is

$$A^{-1} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 1 & -1 & -1 \\ -5/2 & 2 & 7/2 \end{bmatrix}$$

(b) Since A is invertible, then the only solution to the system $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$.

(c) The system $A\vec{x} = \vec{c}$ has only the solution $\vec{x} = A^{-1}\vec{c}$. We can compute this directly:

$$\vec{x} = A^{-1}\vec{c} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 1 & -1 & -1 \\ -5/2 & 2 & 7/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

Q7. Let $A \in M_n(\mathbb{F})$, and suppose that $A^3 + 2A^2 - A + I_n = \mathcal{O}_n$. Prove that A is invertible. What is its inverse?

Solution: We can rearrange the given equation as follows

$$\begin{aligned} A^3 + 2A^2 - A + I_n &= \mathcal{O}_n \\ A^3 + 2A^2 - A &= -I_n \\ (A^2 + 2A - I_n)A &= -I_n \\ -(A^2 + 2A - I_n)A &= I_n \end{aligned}$$

Thus we can see that A has a left inverse. Since A is a square matrix, we know that left invertibility implies invertibility, so A is invertible with inverse $A^{-1} = -A^2 - 2A + I_n$.

Q8. For each of the following transformations, determine whether it is linear or not, and then prove it.

(a) The transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ xy \end{bmatrix}$.

(b) The transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x + 2y + z \\ x + z \end{bmatrix}$.

(c) The transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ -x \\ 2 + 3y \end{bmatrix}$.

(d) For a fixed $\vec{z} \in \mathbb{R}^3$, the transformation $T_{\vec{z}} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T_{\vec{z}}(\vec{x}) = \vec{z} \times \vec{x}$.

Solution: (a) Consider $\vec{x} = \begin{bmatrix}1\\1\end{bmatrix}$. Then we have that

$$2T(\vec{x}) = 2\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$$

and

$$T(2\vec{x}) = T\left(\begin{bmatrix}2\\2\end{bmatrix}\right) = \begin{bmatrix}4\\4\end{bmatrix}$$

Thus it follows that $T(2\vec{x}) \neq 2T(\vec{x})$, and so T is not linear.

(b) Let $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ and $y = \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$, and let $c \in \mathbb{R}$. Then we have that

$$\begin{aligned}
T(c\vec{x} + \vec{y}) &= T\left(\begin{bmatrix}cx_1 + y_1\\cx_2 + y_2\\cx_3 + y_3\end{bmatrix}\right)\\
&= \begin{bmatrix}3(cx_1 + y_1) + 2(cx_2 + y_2) + cx_3 + y_3\\ cx_1 + y_1 + cx_3 + y_3\end{bmatrix}\\
&= c\begin{bmatrix}3x_1 + 2x_2 + x_3\\ x_1 + x_3\end{bmatrix} + \begin{bmatrix}3y_1 + y_2 + y_3\\ y_1 + y_3\end{bmatrix}\\
&= cT(\vec{x}) + T(\vec{y})
\end{aligned}$$

Thus it follows that T is linear.

(c) We have that $T(\vec{0}) = \begin{bmatrix}0\\0\\2\end{bmatrix} \neq \vec{0}$, so T is not linear.

(d) Let $\vec{x}, \vec{y} \in \mathbb{R}^3$ and $c \in \mathbb{R}$. Using properties of the cross product, we have that

$$\begin{aligned}
T_{\vec{z}}(c\vec{x} + \vec{y}) &= \vec{z} \times (c\vec{x} + \vec{y})\\
&= \vec{z} \times (c\vec{x}) + \vec{z} \times \vec{y}\\
&= c(\vec{z} \times \vec{x}) + \vec{z} \times \vec{y}\\
&= cT_{\vec{z}}(\vec{x}) + T_{\vec{z}}(\vec{y})
\end{aligned}$$

Thus $T_{\vec{z}}$ is linear.

Q9. (a) Determine the kernel of the linear transformation $T_1 \colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T_1\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}x_1 + 2x_2 + 3x_3\\ x_2 + 2x_3\\ x_3\end{bmatrix}.$$

(b) Is T_1 one-to-one? Justify your answer.

(c) Determine the range of the linear transformation $T_2\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 + x_3 \end{bmatrix}.$$

(d) Is T_2 onto? Justify your answer.

Solution: (a) The kernel of T_1 is the set of all vectors $\vec{x}$ such that $T_1(\vec{x}) = \vec{0}$. If we let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and suppose that $\vec{x} \in \mathrm{Ker}(T_1)$, it follows that

$$\begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ x_2 + 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the third component, we immediately get that $x_3 = 0$. Simplifying, we find

$$\begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Again, from the second component, we have that $x_2 = 0$. If we simplify once more, we find that

$$\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and so it follows that $x_1 = 0$ as well. Thus, we have that $\vec{x} = \vec{0}$. So we get that $\mathrm{Ker}(T_1) \subseteq \{\vec{0}\}$, and since $\vec{0} \in \mathrm{Ker}(T_1)$, then we get $\mathrm{Ker}(T_1) = \{\vec{0}\}$.

(b) Since $\mathrm{Ker}(T_1)$, we know that T_1 must be one-to-one.

(c) Take any vector $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$. Let $\vec{x} = \begin{bmatrix} y_1 \\ 0 \\ y_2 - y_1 \end{bmatrix}$. Then it follows that

$$T_2(\vec{x}) = \begin{bmatrix} y_1 + 0 \\ y_1 + 0 + y_2 - y_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \vec{y}$$

Thus we have that $\vec{y} \in \mathrm{Range}(T_2)$, and so $\mathbb{R}^2 \subseteq \mathrm{Range}(T_2)$. Note we trivially have that $\mathrm{Range}(T_2) \subseteq \mathbb{R}^2$, so it follows that $\mathrm{Range}(T_2) = \mathbb{R}^2$.

(d) Since $\mathrm{Range}(T_2) = \mathbb{R}^2$, we know that T_2 must be onto.

Q10. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose that there exists a map $S : \mathbb{R}^m \to \mathbb{R}^n$ with the property that that

$$S(\vec{x}) \cdot \vec{v} = \vec{x} \cdot T(\vec{v})$$

for all $\vec{x} \in \mathbb{R}^m$ and $\vec{v} \in \mathbb{R}^n$.

(a) Prove that S is a linear transformation.

(b) Since both T and S are linear, we can find matrices A and B such that $T = T_A$ and $S = T_B$. Prove that $B = A^T$.

Solution: (a) Let $\vec{x}, \vec{y} \in \mathbb{R}^m$, $\vec{v} \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Then we have that

$$\begin{aligned} S(c\vec{x} + \vec{y}) \cdot \vec{v} &= (c\vec{x} + \vec{y}) \cdot T(\vec{v}) \\ &= c(\vec{x} \cdot T(\vec{v})) + \vec{y} \cdot T(\vec{v}) \\ &= c(S(\vec{x}) \cdot \vec{v}) + S(\vec{y}) \cdot \vec{v} \\ &= (cS(\vec{x}) + S(\vec{y})) \cdot \vec{v} \end{aligned}$$

Now if for any $\vec{z} \in \mathbb{R}^n$ we let $[\vec{z}]_i$ denote the ith-component of $\vec{z}$, then we have that $[\vec{z}]_i = \vec{z} \cdot \vec{e}_i$. Thus we get

$$[S(c\vec{x} + \vec{y})]_i = (S(c\vec{x} + \vec{y})) \cdot \vec{e}_i = (cS(\vec{x}) + S(\vec{y})) \cdot \vec{e}_i = [cS(\vec{x}) + S(\vec{y})]_i$$

Since all the components are equal, we then have that $S(c\vec{x} + \vec{y}) = cS(\vec{x}) + S(\vec{y})$, and so S is linear.

(b) Let $\vec{x} = \vec{e}_i \in \mathbb{R}^m$, and let $\vec{v} = \vec{e}_j \in \mathbb{R}^n$. We can evaluate both sides of the given expression. First we have

$$\begin{aligned} S(\vec{x}) \cdot \vec{v} &= S(\vec{e}_i) \cdot \vec{e}_j \\ &= (B\vec{e}_i) \cdot \vec{e}_j \\ &= \vec{b}_i \cdot \vec{e}_j \qquad \text{(where } \vec{b}_i \text{ is the } i\text{-th column vector of } B) \\ &= [\vec{b}_i]_j = (B)_{ij} \end{aligned}$$

On the other hand, we also have that

$$\begin{aligned} \vec{x} \cdot T(\vec{v}) &= \vec{e}_i \cdot (A\vec{e}_j) \\ &= \vec{e}_i \cdot \vec{a}_j \\ &= [\vec{a}_j]_i = (A)_{ji} \end{aligned}$$

Since $(B)_{ij} = (A)_{ji} = (A^T)_{ij}$ for all i, j, we have that $B = A^T$.

Q11. Determine, with justification, whether the following sets are subspaces of $\mathbb{R}^4$.

(a) $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_1 + x_2 - x_3 - x_4 = 0 \right\}$.

(b) $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : |x_1| + |x_2| - |x_3| - |x_4| = 0 \right\}$.

(c) $S = \left\{ \begin{bmatrix} x \\ x^2 \\ y \\ y^2 \end{bmatrix} \in \mathbb{R}^4 : x, y \in \mathbb{R} \right\}$.

Solution: (a) Yes, let us prove this. Firstly we can clearly see that $\vec{0} \in S$. Next take any $\vec{x}, \vec{y} \in S$ and $c \in \mathbb{R}$, and let $\vec{z} = c\vec{x} + \vec{y}$. Then

$$\vec{z} = c\vec{x} + \vec{y} = c \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \\ cx_3 + y_3 \\ cx_4 + y_4 \end{bmatrix}$$

Since $\vec{x}, \vec{y} \in S$, we have that $x_1 + x_2 - x_3 - x_4 = y_1 + y_2 - y_3 - y_4 = 0$. From this it follows that

$$\begin{aligned} & z_1 + z_2 - z_3 - z_4 \\ &= (cx_1 + y_1) + (cx_2 + y_2) - (cx_3 + y_3) - (cx_4 + y_4) \\ &= c(x_1 + x_2 - x_3 - x_4) + (y_1 + y_2 - y_3 - y_4) \\ &= c0 + 0 \\ &= 0 \end{aligned}$$

From this it follows that $\vec{z} = c\vec{x} + \vec{y} \in S$. This proves that S is a subspace (Proposition 8.1.4).

(b) No, this is not closed under vector addition. To see this, take the vectors $\vec{x} = [1\ 1\ 1\ 1]^T$ and $\vec{y} = [1\ 1\ -1\ -1]^T$, both of which are clearly in S. However, if we set $\vec{z} = \vec{x} + \vec{y} = [2\ 2\ 0\ 0]^T$, we see that $\vec{z} \notin S$ as

$$|z_1| + |z_2| - |z_3| - |z_4| = 2 + 2 - 0 - 0 = 4 \neq 0.$$

So S is not closed under vector addition and is therefore not a subspace.

(c) No, this is not closed under scalar multiplication. To see this, take $x = 1$ and $y = 1$ to obtaine the vector $\vec{v} = [1\ 1\ 1\ 1]^T \in S$. If we let $c = 2$, then $c\vec{v} = [2\ 2\ 2\ 2]^T$. If this were in S there would need to be an x such that $x = 2$ and $x^2 = 2$. This is clearly impossible, so $c\vec{v} \notin S$. Thus S is not closed under scalar multiplication and is therefore not a subspace.

Q12. Let V and W be subspaces of $\mathbb{R}^n$. Prove or disprove the following.

(a) $V \cap W$ is a subspace of $\mathbb{R}^n$.

(b) $V \cup W$ is a subspace of $\mathbb{R}^n$.

(c) V^c (the complement of V in $\mathbb{R}^n$) is a subspace of $\mathbb{R}^n$.

Solution: (a) Firstly, we have that $\vec{0} \in V$ and $\vec{0} \in W$, so it follows that $\vec{0} \in V \cap W$. Next we take $\vec{x}, \vec{y} \in V \cap W$, and $c \in \mathbb{R}$. Since V is a subspace and $\vec{x}, \vec{y} \in V$, it follows by Proposition 8.1.4 that $c\vec{x} + \vec{y} \in V$. Similarly, since W is a subspace, we also have that $c\vec{x} + \vec{y} \in W$. Thus $c\vec{x} + \vec{y} \in V \cap W$. So by Proposition 8.1.4 it follows that $V \cap W$ is also a subspace.

(b) Consider the subspaces $V = \mathrm{Span}\{[1\ 0]^T\}$ and $W = \mathrm{Span}\{[0\ 1]^T\}$ as subspaces of $\mathbb{R}^2$. We know these are subspaces as the span of any vectors is always a subspace. Now consider $\vec{x} = [1\ 0]^T \in V$ and $\vec{y} = [0\ 1]^T \in W$. It follows that $\vec{x}, \vec{y} \in V \cup W$. However we can clearly see that $\vec{x} + \vec{y} = [1\ 1]^T$ is neither in V nor in W. Thus $\vec{x} + \vec{y} \notin V \cup W$, and so $V \cup W$ is not closed under vector addition. Thus $V \cup W$ is not a subspace.

(c) Consider $V = \mathbb{R}^3$ as a subspace of $\mathbb{R}^3$. Clearly we then have that $V^c = \varnothing$ (the empty set). However a subspace must be non-empty, so V^c is therefore not a subspace.

Q13. Determine, with justification, whether the following sets are linearly independent.

(a) $S_1 = \left\{ \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} -6 \\ -9 \\ 3 \end{bmatrix} \right\}$.

(b) $S_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$.

(c) $S_3 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$.

Solution: (a) We know for a two element set, that it is linearly independent if and only if neither vector is a scalar multiple of the other. In this case, a simple observation shows us that

$$\begin{bmatrix} -6 \\ -9 \\ 3 \end{bmatrix} = \frac{-3}{2} \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix}.$$

So one vector is a scalar multiple of the other, and hence S_1 is linearly dependent.

(b) By Proposition 8.3.6, we know this set is linearly independent if and only if the number of pivots in the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

is equal to the number of vectors (3 in this case). If we perform row reduction on A, we get

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \begin{matrix} R_2 \to -R_1 + R_2 \\ R_3 \to -2R_1 + R_3 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to \frac{-1}{2} R_3$$

$$\to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 \to -2R_3 + R_1$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 \to -R_2 + R_1$$

This has 3 pivots, and so S_2 is indeed linearly independent.

(c) By Proposition 8.3.6, we know this set is linearly independent if and only if the number of pivots in the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

is equal to the number of vectors (3 in this case). If we perform row reduction on A, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \quad \begin{matrix} R_2 \to -4R_1 + R_2 \\ R_3 \to -7R_1 + R_3 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \to \frac{-1}{3} R_2 \\ R_3 \to \frac{-1}{6} R_3 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to -R_2 + R_3$$

$$\to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 \to -2R_2 + R_1$$

This has only 2 pivots, and so S_3 is indeed linearly dependent.
[Can you figure out how to write one of the vectors of S_3 as a linear combination of the others?]

Q14. Using Proposition 8.3.2 from the notes, we can rewrite Question 2b from Written Assignment 7 as the following lemma (do **not** prove this):

Lemma. *Let $A \in M_n(\mathbb{F})$. If $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of A with distinct eigenvalues λ_1 and λ_2, then the set $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.*

Now let $A \in M_n(\mathbb{F})$, and let $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ be eigenvectors of A with *pairwise distinct* eigenvalues λ_1, λ_2 and λ_3 (in other words, $\lambda_i \neq \lambda_j$ for all $i \neq j$). Prove that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent. You may use Lemma 1 in your proof.

Solution: Let $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ be eigenvectors of A with eigenvalues λ_1, λ_2 and λ_3. In order to show linear independence of the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, consider the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \tag{2}$$

for some $c_1, c_2, c_3 \in \mathbb{F}$. If we multiply equation (1) by the matrix A on the left, we get that

$$\begin{aligned}\vec{0} &= A(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3)\\ &= c_1A\vec{v}_1 + c_2A\vec{v}_2 + c_3A\vec{v}_3\\ &= c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + c_3\lambda_3\vec{v}_3\end{aligned}$$

The last equality uses the fact that $\vec{v}_i$ are eigenvectors, so that $A\vec{v}_i = \lambda_i\vec{v}_i$. Now, we can also multiply equation (1) by the scalar λ_1, and in this case we get

$$\begin{aligned}\vec{0} &= \lambda_1(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3)\\ &= c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2 + c_3\lambda_1\vec{v}_3\end{aligned}$$

Comparing these two equations we get

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + c_3\lambda_3\vec{v}_3 = \vec{0} = c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2 + c_3\lambda_1\vec{v}_3$$

and so

$$c_2(\lambda_2 - \lambda_1)\vec{v}_2 + c_3(\lambda_3 - \lambda_1)\vec{v}_3 = \vec{0}$$

By Lemma 1, we lnow that $\vec{v}_2$ and $\vec{v}_3$ are linearly independent, and so the coefficients of the above equation must be zero. Thus we get $c_2(\lambda_2 - \lambda_1) = 0$ and $c_3(\lambda_3 - \lambda_1) = 0$. Since eigenvectors are distinct, we know that $\lambda_2 - \lambda_1 \neq 0$, and so it follows that $c_2 = 0$. Similarly we get $c_3 = 0$. Plugging this back into equation (1), we get

$$c_1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = \vec{0}$$

and so

$$c_1\vec{v}_1 = \vec{0}$$

Since eigenevectors are non-zero, we must have $c_1 = 0$. So all the coefficients must be zero, and so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Q15. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Prove that the following are equivalent:

(i) T is injective.

(ii) For all vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$, if the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \subseteq \mathbb{R}^n$ is linearly independent, then the corresponding set $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\} \subseteq \mathbb{R}^m$ is also linearly independent.

Solution: Let us prove both directions.

(i) $\implies$ (ii) Suppose T is injective.
Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \subseteq \mathbb{R}^n$ be a linearly indpendent set. Consider the equation

$$c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \ldots + c_kT(\vec{v}_k) = \vec{0} \tag{3}$$

Using linearity of T, we get that

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k) = \vec{0}$$

Since T is injective, we know that $\mathrm{Ker}(T) = \{\vec{0}\}$. So the above equation implies that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}$$

Now since $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent, the above equation implies that $c_1 = c_2 = \ldots = c_k = 0$. But this means that the only solution to equation (2) is the trivial equation. So it follows that the set $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$ is linearly independent.

(ii) $\implies$ (i)] Suppose that we have that for every linearly independent set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \subseteq \mathbb{R}^n$, the corresponding set $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$ is also linearly independent.
We can show T is injective by showing its kernel contains only the zero vector. So take $\vec{x} \in \mathrm{Ker}(T)$, so $T(\vec{x}) = \vec{0}$. Suppose that $\vec{x} \neq \vec{0}$. Then it follows that the set $\{\vec{x}\}$ is linearly independent. By assumption this means that $\{T(\vec{x})\}$ is linearly independent, which means that $T(\vec{x}) \neq \vec{0}$. This contradicts the assumption that $\vec{x} \in \mathrm{Ker}(T)$.
So it follows that if $\vec{x} \in \mathrm{Ker}(T)$, then $\vec{x} = \vec{0}$. Thus we have $\mathrm{Ker}(T) = \{\vec{0}\}$, and so T is injective.

Q16. Determine whether the matrix

$$A = \begin{bmatrix} 3 & 2 & -2 & -2 \\ -1 & 2 & 2 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

is diagonalizable over $\mathbb{R}$ and/or $\mathbb{C}$.

Solution: First we find the eigenvalues of A by calculating $\det(A - \lambda I_4)$. We have

$$
\begin{aligned}
C_A(\lambda) &= \det \begin{bmatrix} 3-\lambda & 2 & -2 & -2 \\ -1 & 2-\lambda & 2 & 0 \\ 1 & 0 & -1-\lambda & 0 \\ 0 & 2 & 1 & -\lambda \end{bmatrix} \\
&= -(-2)\det \begin{bmatrix} -1 & 2-\lambda & 2 \\ 1 & 0 & -1-\lambda \\ 0 & 2 & 1 \end{bmatrix} + (-\lambda)\det \begin{bmatrix} 3-\lambda & 2 & -2 \\ -1 & 2-\lambda & 2 \\ 1 & 0 & -1-\lambda \end{bmatrix} \\
&= 2\left(-1\det \begin{bmatrix} 2-\lambda & 2 \\ 2 & 1 \end{bmatrix} - (-1-\lambda)\det \begin{bmatrix} -1 & 2-\lambda \\ 0 & 2 \end{bmatrix}\right) \\
&\quad - \lambda\left(1\det \begin{bmatrix} 2 & -2 \\ 2-\lambda & 2 \end{bmatrix} + (-1-\lambda)\det \begin{bmatrix} 3-\lambda & 2 \\ -1 & 2-\lambda \end{bmatrix}\right) \\
&= 2\left[(-1)(2-\lambda-4) + (1+\lambda)(-2)\right] \\
&\quad - \lambda\left[(4-(-2)(2-\lambda) - (1+\lambda)((3-\lambda)(2-\lambda)+2)\right] \\
&= 2\left[\lambda+2-2-2\lambda\right] - \lambda\left[4+4-2\lambda-(1+\lambda)(\lambda^2-5\lambda+8)\right] \\
&= -2\lambda - \lambda\left[8-2\lambda-(\lambda^2-5\lambda+8)-(\lambda^3-5\lambda^2+8\lambda)\right] \\
&= -2\lambda - \lambda\left[8-2\lambda-\lambda^2+5\lambda-8-\lambda^3+5\lambda^2-8\lambda\right] \\
&= -2\lambda - \lambda\left[(8-8)+(-2+5-8)\lambda+(-1+5)\lambda^2-\lambda^3\right] \\
&= \lambda\left[\lambda^3-4\lambda^2+5\lambda-2\right] \\
&= \lambda(\lambda-1)\left[\lambda^2-3\lambda+2\right] \\
&= \lambda(\lambda-1)^2(\lambda-2)
\end{aligned}
$$

So A has eigenvalues $0, 1, 2$ with algebraic multiplicities $a_0 = 1$, $a_1 = 2$, $a_2 = 1$.
Since the characteristic polynomial has no irreducible factors, A is diagonalisable (over both $\mathbb{R}$ and $\mathbb{C}$) if and only if the geometric multiplicities agree with the algebraic multiplicities.
Since $1 \leq g_{\lambda_i} \leq a_{\lambda_i}$, we immediately know that $g_0 = g_2 = 1$. So all that is left to do is to check the value of g_1. To do this we need to find the eigenspace.

We have

$$\begin{aligned}
E_2 &= \mathrm{Null}(A - 2I_4)\\
&= \mathrm{Null}\left(\begin{bmatrix} 1 & 2 & -2 & -2\\ -1 & 0 & 2 & 0\\ 1 & 0 & -3 & 0\\ 0 & 2 & 1 & -2 \end{bmatrix}\right)\\
&= \mathrm{Null}\left(\begin{bmatrix} 1 & 2 & -2 & -2\\ 0 & 2 & 0 & -2\\ 0 & -2 & -1 & 2\\ 0 & 2 & 1 & -2 \end{bmatrix}\right)\\
&= \mathrm{Null}\left(\begin{bmatrix} 1 & 2 & -2 & -2\\ 0 & 1 & 0 & -1\\ 0 & -2 & -1 & 2\\ 0 & 2 & 1 & -2 \end{bmatrix}\right)\\
&= \mathrm{Null}\left(\begin{bmatrix} 1 & 2 & -2 & -2\\ 0 & 1 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & 0 & 1 & 0 \end{bmatrix}\right)\\
&= \mathrm{Null}\left(\begin{bmatrix} 1 & 2 & -2 & -2\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}\right)\\
&= \mathrm{Null}\left(\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}\right)
\end{aligned}$$

Now we can calculate g_2 as

$$g_2 = \dim(E_2) = \mathrm{nullity}\left(\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = 4 - \mathrm{rank}\left(\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = 4-3 = 1$$

Since $g_2 = 1 \neq a_2$, it follows that A is not diagonalisable.

Q17. Let $A \in M_4(\mathbb{C})$ and suppose it is diagonalizable over $\mathbb{C}$. Furthermore suppose it has eigenvalues $\lambda = 1, 2, 3$, and that these are all its eigenvalues. How many diagonal matrices D are there which are similar to A?

Solution: We know that there are three distinct eigenvalues, and the sum of their (algebraic) multiplicities is 4. So at least one of these eigenvalues is repeated, and has a multiplicity of 2.

Let $\lambda_1, \lambda_2, \lambda_3$ be these eigenvalues, such that λ_1 is the repeated eigenvalue. We know that if D is a diagonal matrix similar to A, then the entries of D must be the these eigenvalues (and in particular λ_1 appear twice). So this gives us the possibilities for D:

(1) $D = \text{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_3)$.
(2) $D = \text{diag}(\lambda_1, \lambda_1, \lambda_3, \lambda_2)$.
(3) $D = \text{diag}(\lambda_1, \lambda_2, \lambda_1, \lambda_3)$.
(4) $D = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_1)$.
(5) $D = \text{diag}(\lambda_1, \lambda_3, \lambda_1, \lambda_2)$.
(6) $D = \text{diag}(\lambda_1, \lambda_3, \lambda_2, \lambda_1)$.
(7) $D = \text{diag}(\lambda_2, \lambda_1, \lambda_1, \lambda_3)$.
(8) $D = \text{diag}(\lambda_2, \lambda_1, \lambda_3, \lambda_1)$.
(9) $D = \text{diag}(\lambda_2, \lambda_3, \lambda_1, \lambda_1)$.
(10) $D = \text{diag}(\lambda_3, \lambda_1, \lambda_1, \lambda_2)$.
(11) $D = \text{diag}(\lambda_3, \lambda_1, \lambda_2, \lambda_1)$.
(12) $D = \text{diag}(\lambda_3, \lambda_2, \lambda_1, \lambda_1)$.

This gives us 12 possibilities for D.

A more combinatorial approach would involve counting the number of permutation of 4 elements (which a total of $4! = 24$ permutations), and then dividing by 2, since we can interchange the position of the two λ_1 entries. This gives us a total of $4!/2! = 24/2 = 12$ combinations, and agrees with what we have found above.

Q18. Consider $P_n(\mathbb{F})$, the vector space of polynomials over $\mathbb{F}$ of degree at most n. Define the function $D : P_n(\mathbb{F}) \to P_n(\mathbb{F})$ as follows: for a polynomial $p(x) = \sum_{i=0}^n a_i x^i$ where $a_i \in \mathbb{F}$, set

$$Dp(x) = \sum_{i=1}^n i a_i x^{i-1}$$

(a) Show that D is a linear map.
(b) What is the range of D? Is D surjective?
(c) What is the kernel of D? Is D injective?

Solution: (a) To show that D is linear, let $p, q \in P_n(\mathbb{F})$ and let $c \in \mathbb{F}$. Let us write $p(x) = \sum_{i=0}^n a_i x^i$ and $q(x) = \sum_{i=0}^n b_i x^i$. First note that

$$(cp + q)(x) = \sum_{i=0}^n (ca_i + b_i) x^i$$

$$\begin{aligned} D(cp+q)(x) &= \sum_{i=1}^{n} i(ca_i + b_i)x^{i-1} \\ &= \sum_{i=1}^{n} (cia_i + ib_i)x^{i-1} \\ &= c\sum_{i=1}^{n} ia_i x^{i-1} + \sum_{i=1}^{n} ib_i x^{i-1} \\ &= cD(p)(x) + D(q)(x) \end{aligned}$$

So it follows that $D(cp+q) = cD(p) + D(q)$, and so D is linear.

(b) We claim that $\text{Range}(D) = P_{n-1}(\mathbb{F})$. Indeed we have that for any $p \in P_n(\mathbb{F})$, we have that $D(p)$ is a polynomial of degree at most $n-1$. So this means that $\text{Range}(D) \subseteq P_{n-1}(\mathbb{F})$.
For the other direction, take any polynomial $q \in P_{n-1}$ and write its coefficients as $q(x) = \sum_{i=0}^{n-1} b_i x^i$. Now for $i \geq 1$, set $a_i = b_{i-i}/i$ and $a_0 = 0$. Furthermore let $p(x) = \sum_{i=0}^{n} a_i x^i$. Then we have that

$$\begin{aligned} Dp(x) &= \sum_{i=1}^{n} ia_i x^{i-1} \\ &= \sum_{i=1}^{n} i\frac{b_{i-1}}{i} x^{i-1} \\ &= \sum_{i=1}^{n} b_{i-1} x^{i-1} \\ &= \sum_{j=0}^{n-1} b_j x^j = q(x) \end{aligned}$$

This means that $Dp(x) = q(x)$, and so it follows that $q \in \text{Range}(D)$. Hence $P_{n-1}(\mathbb{F}) \subseteq \text{Range}(D)$, and so we have equality.
Since $\text{Range}(D) \neq P_n(\mathbb{F})$, this means that D is not surjective.

(c) We claim that $\text{Ker}(D) = P_0(\mathbb{F})$, i.e. the constant polynomials. If we take any constant polynomial $p(x) = \sum_{i=0}^{n} a_i x^i$, then since it is constant, this means that $a_i = 0$ for all $i \neq 0$. So we have that

$$Dp(x) = \sum_{i=1}^{n} ia_i x^{i-1} = \sum_{i=1}^{n} 0x^{i-1} = 0$$

and so $p \in \text{Ker}(D)$.
For the other direction, if we take $p \in \text{Ker}(D)$, then this means that $Dp = 0$. If we write $p(x) = \sum_{i=0}^{n} a_i x^i$, then we have

$$Dp(x) = \sum_{i=1}^{n} ia_i x^{i-1} = 0$$

So it follows that $a_i = 0$ for all $i \geq 1$. This means that p is a constant polynomial, thus showing that $\text{Ker}(D) = P_0(\mathbb{F})$.
Since $\text{Ker}(D) \neq \{0\}$, this means that D is not injective.

Q19. Consider the plane $\mathcal{P}$ in $\mathbb{R}^3$ with scalar equation $3x - y + 2z = 0$. For each of the following lines, find all intersection points with $\mathcal{P}$, and describe what this set looks like geometrically (eg. "a line through $\vec{u}$ with direction vector $\vec{v}$").

(a) $\mathcal{L}_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

(b) $\mathcal{L}_2 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} : t \in \mathbb{R} \right\}$

Solution: (a) For any vector $\vec{v} \in \mathcal{L}_1$, we can write it as $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ 1+t \\ 2+t \end{bmatrix}$ for some $t \in \mathbb{R}$. This vector will lie on the plane precisely when it satisfies the scalar equation $3x - y + 2z = 0$. Plugging this in, we get

$$\begin{aligned} 3t - (1+t) + 2(2+t) &= 0 \\ 4t + 3 &= 0 \\ t &= \frac{-3}{4} \end{aligned}$$

Using this we get that $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix}$. Geometrically this a single point (namely the one at $\frac{1}{4} \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix}$).

(b) For any vector $\vec{v} \in \mathcal{L}_2$, we can write it as $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -1+3t \\ 1+t \\ 2-4t \end{bmatrix}$ for some $t \in \mathbb{R}$. This vector will lie on the plane precisely when it satisfies the scalar equation $3x - y + 2z = 0$. Plugging this in, we get

$$3(-1+3t) - (1+t) + 2(2-4t) = (9-1-8)t + (-3-1+4) \qquad = 0$$

This tells us that the every vector in $\mathcal{L}_2$ is in the plane. In other words the intersection set is $\mathcal{L}_2$ itself. Geometrically, this is (clearly) the line through $[-1\ 1\ 2]^T$ with direction vector $[3\ 1\ -4]^T$.

Q20. (8 marks) Let $\mathcal{P}$ be a plane in $\mathbb{R}^n$ through the origin with direction vectors $\vec{u}$ and $\vec{v}$. Assume moreover that $\vec{u}$ and $\vec{v}$ are orthogonal and are unit vectors. We will define the projection of a vector $\vec{z}$ onto $\mathcal{P}$ by

$$\text{proj}_{\mathcal{P}}(\vec{z}) = \text{proj}_{\vec{u}}(\vec{z}) + \text{proj}_{\vec{v}}(\vec{z})$$

(a) (2 marks) Prove that $\text{proj}_{\mathcal{P}}$ is linear. That is, prove that $\text{proj}_{\mathcal{P}}(a\vec{x}+b\vec{y}) = a\,\text{proj}_{\mathcal{P}}(\vec{x})+b\,\text{proj}_{\mathcal{P}}(\vec{y})$ for all $a, b \in \mathbb{R}$ and $\vec{x}, \vec{y} \in \mathbb{R}^n$.

(b) (3 marks) Show that if $\vec{w} \in \mathcal{P}$, then $\text{proj}_{\mathcal{P}}(\vec{w}) = \vec{w}$.

(c) (3 marks) Use the previous result to show that if $\vec{x} \in \mathbb{R}^n$ and $\vec{w} \in \mathcal{P}$, then $\text{proj}_{\mathcal{P}}(\vec{x})\cdot\vec{w} = \vec{x} \cdot \vec{w}$.

Solution: (a) Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

$$\begin{aligned}
\text{proj}_{\mathcal{P}}(a\vec{x} + b\vec{y}) &= \text{proj}_{\vec{u}}(a\vec{x} + b\vec{y}) + \text{proj}_{\vec{v}}(a\vec{x} + b\vec{y}) \\
&= (\vec{u} \cdot [a\vec{x} + b\vec{y}])\vec{u} + (\vec{v} \cdot [a\vec{x} + b\vec{y}])\vec{v} \\
&= (\vec{u} \cdot a\vec{x})\vec{u} + (\vec{u} \cdot b\vec{y})\vec{u} + (\vec{v} \cdot a\vec{x})\vec{v} + (\vec{v} \cdot b\vec{y})\vec{v} \\
&= a(\vec{u} \cdot \vec{x})\vec{u} + b(\vec{u} \cdot \vec{y})\vec{u} + a(\vec{v} \cdot \vec{x}) + b(\vec{v} \cdot \vec{y})\vec{v} \\
&= a\,\text{proj}_{\vec{u}}(\vec{x}) + b\,\text{proj}_{\vec{u}}(\vec{y}) + a\,\text{proj}_{\vec{v}}(\vec{x}) + b\,\text{proj}_{\vec{v}}(\vec{y}) \\
&= a\,\text{proj}_{\mathcal{P}}(\vec{x}) + b\,\text{proj}_{\mathcal{P}}(\vec{y})
\end{aligned}$$

as required.

(b) Let $\vec{w} \in \mathcal{P}$. Then we can find $a, b \in \mathbb{R}$ such that $\vec{w} = a\vec{u} + b\vec{v}$. We can then compute

$$\begin{aligned}
\text{proj}_{\mathcal{P}}(\vec{w}) &= \text{proj}_{\mathcal{P}}(a\vec{u} + b\vec{v}) \\
&= a\,\text{proj}_{\mathcal{P}}(\vec{u}) + b\,\text{proj}_{\mathcal{P}}(\vec{v}) \\
&= a\,\text{proj}_{\vec{u}}(\vec{u}) + a\,\text{proj}_{\vec{v}}(\vec{u}) + b\,\text{proj}_{\vec{u}}(\vec{v}) + b\,\text{proj}_{\vec{v}}(\vec{v})
\end{aligned}$$

Now, it is not hard to verify that

$$\text{proj}_{\vec{u}}(\vec{u}) = \vec{u} \qquad \text{proj}_{\vec{v}}(\vec{v}) = \vec{v}$$

Moreover, since $\vec{u}$ and $\vec{v}$ are orthogonal, we have that

$$\text{proj}_{\vec{u}}(\vec{v}) = \text{proj}_{\vec{v}}(\vec{u}) = \vec{0}$$

And so we have that

$$\begin{aligned}
\text{proj}_{\mathcal{P}}(\vec{w}) &= a\,\text{proj}_{\vec{u}}(\vec{u}) + a\,\text{proj}_{\vec{v}}(\vec{u}) + b\,\text{proj}_{\vec{u}}(\vec{v}) + b\,\text{proj}_{\vec{v}}(\vec{v}) \\
&= a\vec{u} + \vec{0} + \vec{0} + b\vec{v} \\
&= \vec{w}
\end{aligned}$$

as required.

(c) Let $\vec{w} \in \mathcal{P}$ and $\vec{x} \in \mathbb{R}^n$. Then

$$\begin{aligned}
\text{proj}_{\mathcal{P}}(\vec{x}) \cdot \vec{w} &= [(\vec{x} \cdot \vec{u})\vec{u} + (\vec{x} \cdot \vec{v})\vec{v}] \cdot \vec{w} \\
&= (\vec{x} \cdot \vec{u})(\vec{u} \cdot \vec{w}) + (\vec{x} \cdot \vec{v})(\vec{v} \cdot \vec{w}) \\
&= \vec{x} \cdot [\vec{u}(\vec{u} \cdot \vec{w}) + \vec{v}(\vec{v} \cdot \vec{w})] \\
&= \vec{x} \cdot \text{proj}_{\mathcal{P}}(\vec{w}) \\
&= \vec{x} \cdot \vec{w}
\end{aligned}$$

as required.

Q21. Let

$$\vec{x} = \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & -2 \\ -1 & -3 \end{bmatrix}, C = \begin{bmatrix} 0 & -2 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Compute the following expressions if they exist. For each expression, if it is not valid write 'NO', otherwise if it is valid write 'YES' **and** compute it.

(i) $\vec{x} - 3\vec{y}$ (ii) $\|\vec{x}\|$ (iii) $\vec{x} \cdot \vec{y} \cdot \vec{y}$ (iv) $\text{proj}_{\vec{x}}(\vec{y})$

(v) $B\vec{x}$ (vi) $C\vec{x}$ (vii) $A + 2B$ (viii) $2A - C^T$

(ix) AB (x) BA (xi) $\det(B)$ (xii) $\det(C)$

Solution:

(i) $\vec{x} - 3\vec{y} = \begin{bmatrix} -4 \\ -7 \\ 7 \end{bmatrix}$ (ii) $\|\vec{x}\| = 6$

(iii) NO (iv) $\text{proj}_{\vec{x}}(\vec{y}) = \dfrac{-1}{9} \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix}$

(v) NO (vi) $C\vec{x} = \begin{bmatrix} 12 \\ -10 \end{bmatrix}$

(vii) NO (viii) $2A - C^T = \begin{bmatrix} 2 & 3 \\ 6 & -7 \\ -1 & 7 \end{bmatrix}$

(ix) $AB = \begin{bmatrix} 3 & -5 \\ 10 & 2 \\ -4 & -12 \end{bmatrix}$ (x) NO

(xi) $\det(B) = -14$ (xii) NO

Q22. Let $A \in M_3(\mathbb{R})$ be the following matrix.

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 7 & -5 \\ -2 & -3 & 2 \end{bmatrix}.$$

(a) Find A^{-1}. Show your working, including labelling row operations, if any.

(b) Consider the following system of equations

$$\begin{aligned} x_1 + 3x_2 - 2x_3 &= a \\ 2x_1 + 7x_2 - 5x_3 &= b \\ -2x_1 - 3x_2 + 2x_3 &= c \end{aligned}$$

for some $a, b, c \in \mathbb{R}$. Why is this system always consistent?

(c) Give a solution to this system in terms of a, b, and c.

(d) Find an expression for a matrix B with eigenpairs $\left(-1, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}\right)$, $\left(1, \begin{bmatrix} 3 \\ 7 \\ -3 \end{bmatrix}\right)$, and $\left(1, \begin{bmatrix} -2 \\ -5 \\ 2 \end{bmatrix}\right)$. You do not have to simplify your answer.

Solution: (a) We can compute the inverse as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 7 & -5 & 0 & 1 & 0 \\ -2 & -3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 3 & -2 & 2 & 0 & 1 \end{array}\right] \quad \begin{array}{c} R_2 \to -2R_1 + R_2 \\ R_3 \to 2R_1 + R_3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 8 & -3 & 1 \end{array}\right] \quad R_3 \to 3R_2 + R_3$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 17 & -6 & 2 \\ 0 & 1 & 0 & 6 & -2 & 1 \\ 0 & 0 & 1 & 8 & -3 & 1 \end{array}\right] \quad \begin{array}{c} R_1 \to 2R_3 + R_1 \\ R_2 \to R_3 + R_2 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 6 & -2 & 1 \\ 0 & 0 & 1 & 8 & -3 & 1 \end{array}\right] \quad R_1 \to -3R_2 + R_1$$

So we get that

$$A^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 6 & -2 & 1 \\ 8 & -3 & 1 \end{bmatrix}$$

(b) Since A is invertible, it has rank 3 (which is the number of rows). So by the system rank theorem this system is consistent for all choices of $a, b, c \in \mathbb{R}$.

(c) If we let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then we can write this system as $A\vec{x} = \vec{w}$.
Since we know the inverse of A, we have that

$$\vec{x} = A^{-1}\vec{w} = \begin{bmatrix} -1 & 0 & -1 \\ 6 & -2 & 1 \\ 8 & -3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -a-c \\ 6a-2b+c \\ 8a-3b+c \end{bmatrix}$$

(d) Notice the eigenvectors given are the columns of A. This means that if we diagonalise B, then we get

$$B = ADA^{-1}$$

where $D = \text{diag}(-1, 1, 1)$. Writing this in full we have

$$B = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 7 & -5 \\ -2 & -3 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 6 & -2 & 1 \\ 8 & -3 & 1 \end{bmatrix}$$

If we were to complete this computation (which is not necessary) we would find

$$B = \begin{bmatrix} 3 & 0 & 2 \\ 4 & 1 & 4 \\ -4 & 0 & -3 \end{bmatrix}$$

Q23. Let $\mathcal{P}$ be a plane in $\mathbb{R}^3$ through the origin with normal vector $\vec{n}$. Suppose that $\mathcal{L}$ is a line (also in $\mathbb{R}^3$) going through $\vec{u}$ with direction vector $\vec{v}$. Suppose further that $\vec{n} \cdot \vec{v} = 0$.

(a) Write down a vector equation for the line $\mathcal{L}$.

(b) Prove that if $\vec{u} \in \mathcal{P}$ then $\mathcal{L}$ is contained in $\mathcal{P}$.

Solution: (a) One vector equation for $\mathcal{L}$ is

$$\vec{\ell} = \vec{u} + t\vec{v}$$

for all $t \in \mathbb{R}$.

(b) Let $\vec{u} \in \mathcal{P}$. Since $\vec{n}$ is a normal vector for $\mathcal{P}$, this means that $\vec{n} \cdot \vec{p} = 0$ if and only if $\vec{p} \in \mathcal{P}$. In particular we have that $\vec{u} \cdot \vec{n} = 0$.
Now take any $\vec{\ell} \in \mathcal{L}$. By the above vector equation, we can write $\vec{\ell} = \vec{u} + t\vec{v}$ for some $t \in \mathbb{R}$. It follows that

$$\vec{\ell} \cdot \vec{n} = (\vec{u} + t\vec{v}) \cdot \vec{n} = (\vec{u} \cdot \vec{n}) + t(\vec{v} \cdot \vec{n}) = 0 + t0 = 0$$

So we have that $\vec{\ell} \cdot \vec{n} = 0$, and so it follows that $\vec{\ell} \in \mathcal{P}$. Hence $\mathcal{L}$ is contained in $\mathcal{P}$.

Q24. Recall that the **trace** of a matrix is the sum of the diagonal entries. Let $A, B \in M_n(\mathbb{F})$ with entries

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}.$$

(a) Express $\mathrm{tr}(A)$ in terms of the entries of A.

(b) Express the (i, j)-th entry of matrix product AB, in terms of the entries of A and B.

(c) Use the previous two results to find an expression for $\mathrm{tr}(AB)$ in terms of the entries of A and B.

(d) Use part (c) to show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(e) Use part (d) to show that if $S, T \in M_n(\mathbb{F})$ are similar matrices, then $\mathrm{tr}(S) = \mathrm{tr}(T)$.

Solution: (a) Applying the definition we get

$$\mathrm{tr}(A) = \sum_{k=1}^{n} a_{kk}$$

(b) Applying the definition we get

$$(AB)_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}$$

(c) Using the previous parts, we have that

$$\mathrm{tr}(AB) = \sum_{k=1}^{n} (AB)_{kk} = \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} b_{lk}$$

(d) From before we have that

$$\mathrm{tr}(AB) = \sum_{k=1}^{n} (AB)_{kk} = \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} b_{lk}$$

In a similar way we have

$$tr(BA) = \sum_{l=1}^{n} (BA)_{ll} = \sum_{l=1}^{n} \sum_{k=1}^{n} b_{lk} a_{kl}$$

By reordering the summations (and using commutativity of $\mathbb{F}$), we see that

$$\sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} b_{lk} = \sum_{l=1}^{n} \sum_{k=1}^{n} b_{lk} a_{kl}$$

Thus it follows that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(e) Let S, T be similar matrices, so that $S = PTP^{-1}$ for some invertible $P \in M_n(\mathbb{F})$. Let $A = P$ and $B = TP^{-1}$. Then using $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ we have that

$$\operatorname{tr}(S) = \operatorname{tr}(PTP^{-1}) = \operatorname{tr}(AB) = \operatorname{tr}(BA) = \operatorname{tr}(TP^{-1}P) = \operatorname{tr}(T)$$