Problems

1. A metal rod of length L is cooled so that the temperature at both ends of the rod are 0°C. The temperature u (in degrees Celsius) at any point x along the rod at time t is given by the equation
$$u(x,t) = \sin(kx)\, e^{-mt} \tag{1}$$
where k and m are constants.

   (a) Find a non-zero value of k which will satisfy the boundary conditions, that is $u(0,t) = u(L,t) = 0$. [Note: there are multiple correct choices for k, you need only provide one.]

   (b) It is known that temperature behaves according to the so-called heat equation
$$u_{xx}(x,t) = \alpha u_t(x,t)$$
where α is a constant. Show that with an appropriate choice of m, the formula for the rod temperature in (1) satisfies the heat equation.

2. Let $f(x,y) = x^2 + y^2$.

   (a) Sketch the level curves of f.

   (b) Compute the first order partial derivatives of f.

   (c) Find the tangent plane of f at the point (a,b).

   (d) Show that the tangent plane never intersects the graph of f except at the point (a,b).

3. The first order partial derivatives of an unknown differentiable function f are given by the formulae
$$f_x(x,y) = \frac{4x+y}{2x^2+xy} \qquad f_y(x,y) = \frac{1}{2x+y}$$
Furthermore, the variables x and y depend on t in the following way
$$x(t) = t^2 \qquad y(t) = \sin(t)$$
Find $\dfrac{df}{dt}(x(t), y(t))$.

4. Consider the function
$$f(x,y) = \frac{|x|^\alpha |y|}{x^2+y^2}$$
where $\alpha > 0$ is a constant.

   (a) Show that if $\alpha \leq 1$, then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

   (b) Show that if $\alpha \geq 2$, then $\lim_{(x,y)\to(0,0)} f(x,y)$ exists and find its value.