Solution to "Nine Quilts"

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Introduction

This is a solution for the sudoku puzzle "Nine Quilts", available to play at the following url: `https://sudokupad.app/1ywmlyrmdn`. Additionally, you can find it at `https://logic-masters.de/Raetselportal/Raetsel/zeigen.php?id=000UQ1`. Naturally there will be spoilers here, so try the sudoku before reading on! This guide provides two solutions: an abridged version with only a few difficult steps included, and a complete version with the remaining details filled in. While the aim is to be fairly comprehensive, some familiarity with sudoku puzzles will be assumed, and so not every step will be spelled out. Note that we will only present one solution path, there are very likely other approaches!

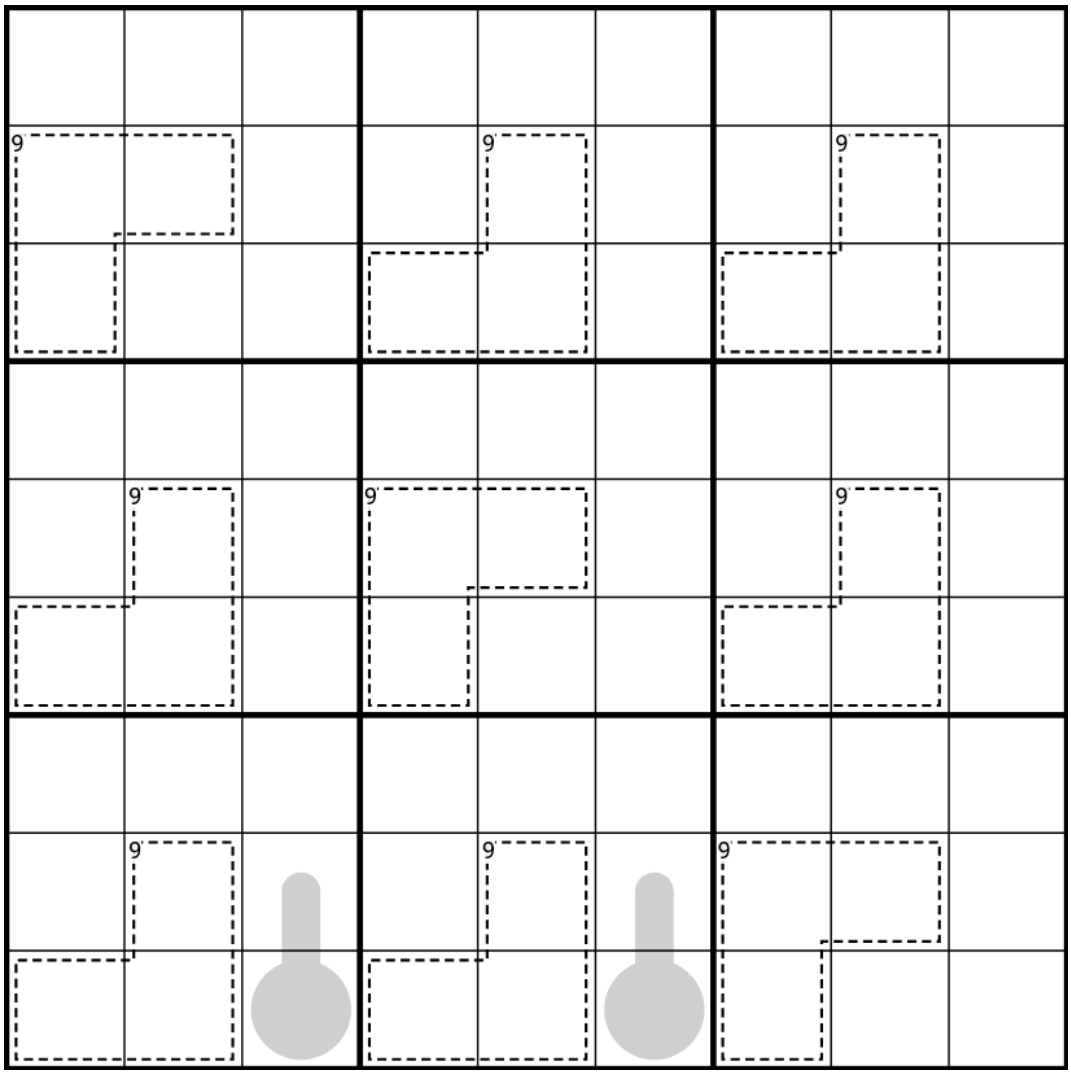
Some quick words on notation: the rows, columns and boxes are numbered in the usual reading order, i.e. from left to right and then top to bottom. Cells will be labelled using 'RxCy' notation. So for instance, the cell in the third row and the fourth column is labelled R3C4.

If you just want to see the final solution, it can be found on the final page.

The Puzzle

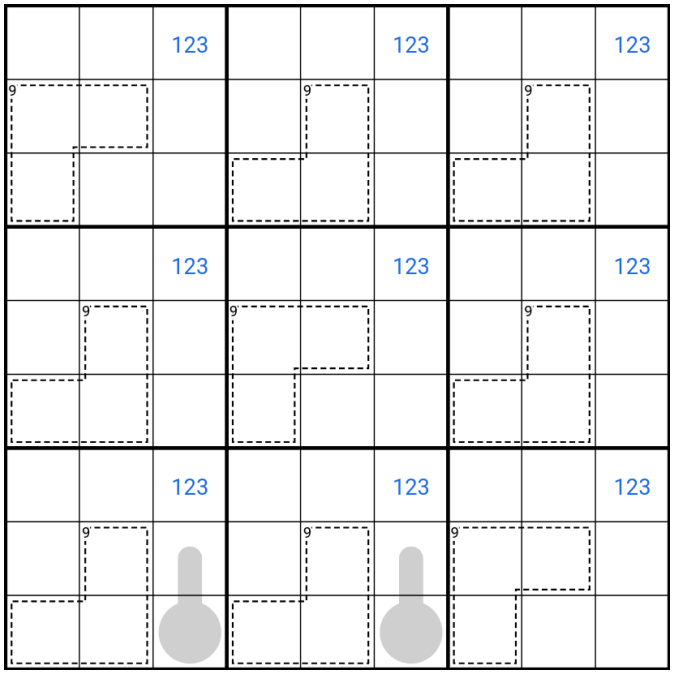
The puzzle grid is below, and the rules are as follows.

- Normal sudoku rules apply: place the digits 1-9 once in each row, column, and box.
- Digits within a cage sum to the number in its upper left-hand corner.
- Digits along a thermometer increase from the bulb end.
- Orthogonally adjacent digits cannot have a difference that is a multiple of 3.

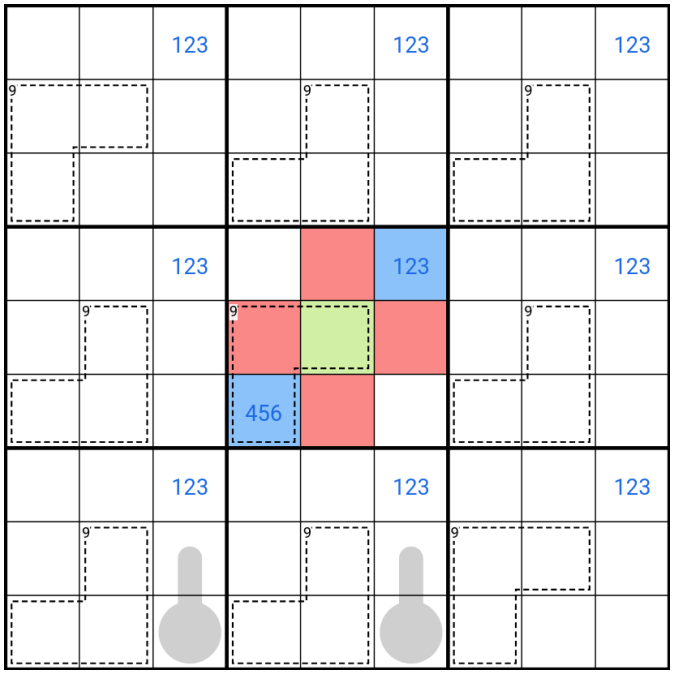


The Abridged Solution

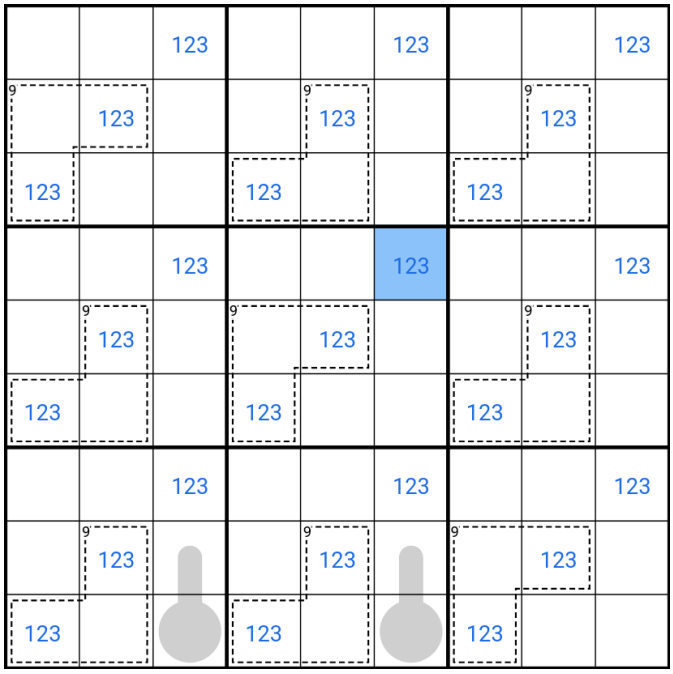
Firstly, notice that the low digits (1, 2, and 3) must appear twice in every cage. This means that say in the bottom two rows, the low digits are always inside a cage. We can repeat this for all relevant rows, and this forces low digits in the upper-right corner of every box.



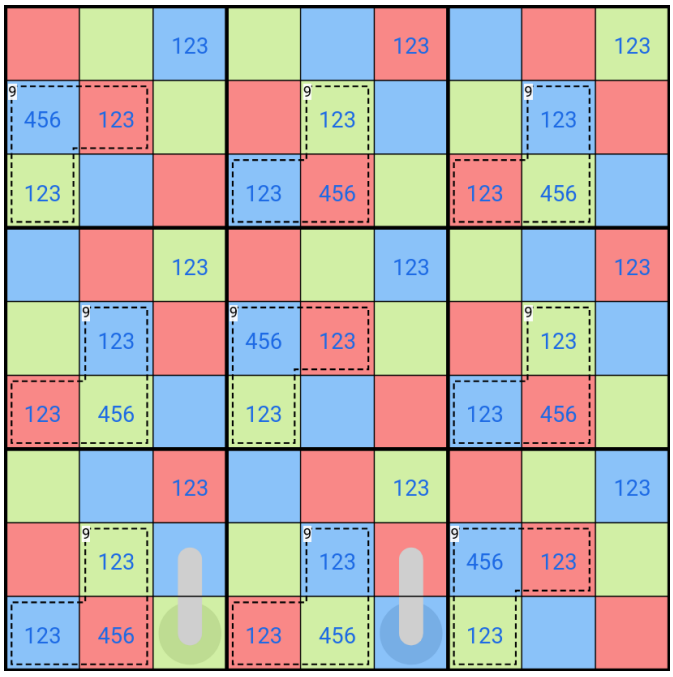
Next we will work with the center box. Denote the modularity classes by colour, and colour the upper-right digit blue. If there was a blue digit in the bottom left corner, this would force a pattern like below, which would not be possible.



Since the blue digit in the cage must be the medium digit (4, 5, or 6), then it follows that the bottom-left digit is also low. We can fill in the grid as below.



Then we an use the colouring rules to fill out the rest of the grid with the appropriate colour.



The thermometers then disambiguate the modularity classes, and the sudoku is solved. Again, the solution can be found on the final page.

The Complete Solution

This puzzle is mostly about solving the "big idea", once you have that, the rest of the solution falls into place fairly quickly. Without it, it becomes a very difficult (perhaps impossible?) puzzle to solve. So, we will spend a little bit of time to introduce these ideas and some terminology, before actually attempting the puzzle.

The key to solving is the final rule: "Orthogonally adjacent digits cannot have a difference that is a multiple of 3". If you have a little bit of mathematics knowledge, you might realise that this is essentially saying that digits which are equivalent modulo 3, cannot be orthogonal. If that last statement didn't make any sense, consider the set of digits $\{1, 4, 7\}$. If I take any two of these, their difference will 3 or 6, so they cannot be next to each other. However if I take one of these digits and any other digit from 1 to 9, then their difference is never 3 or 6, and so they can always be placed next to each other. This also works for the set of digits $\{2, 5, 8\}$ and $\{3, 6, 9\}$. I will call each of these sets a *modular class*, and we will note that each of these three modular classes has exactly three digits.

Next, I will also define some additional terminology. I will call any of the digits 1, 2, or 3 a *low digit*, any of the digits 4, 5, or 6 a *medium digit*, and any of the digits 7, 8, or 9 a *high digit*. Notice that every modular class has exactly one low digit, one medium digit, and one high digit.

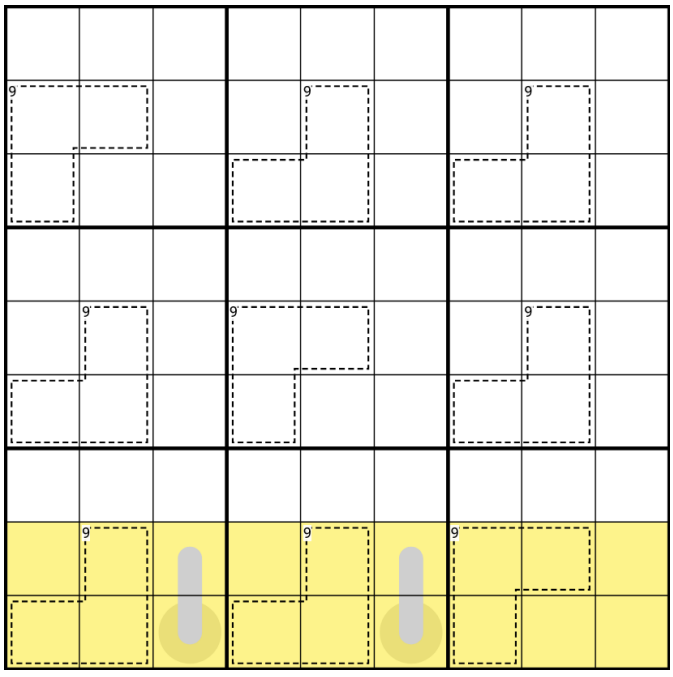
Finally I want to take a look at the killer cages, all of which sum to 9. There are three possible ways to do this with three digits, namely 126, 135, and 234. There are two very important observations about these sums:

1. Each sum has exactly one digit from each of the three modular classes. (This can be checked manually or proved using some modular arithmetic).

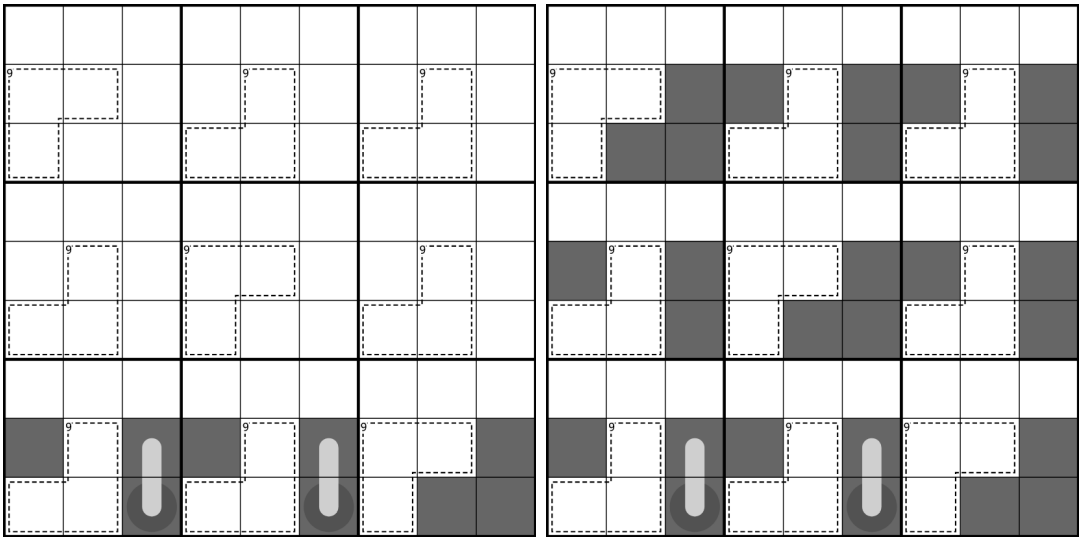
2. Each sum has exactly two low digits, and one medium digit.

With these observations out of the way, we can finally begin solving.

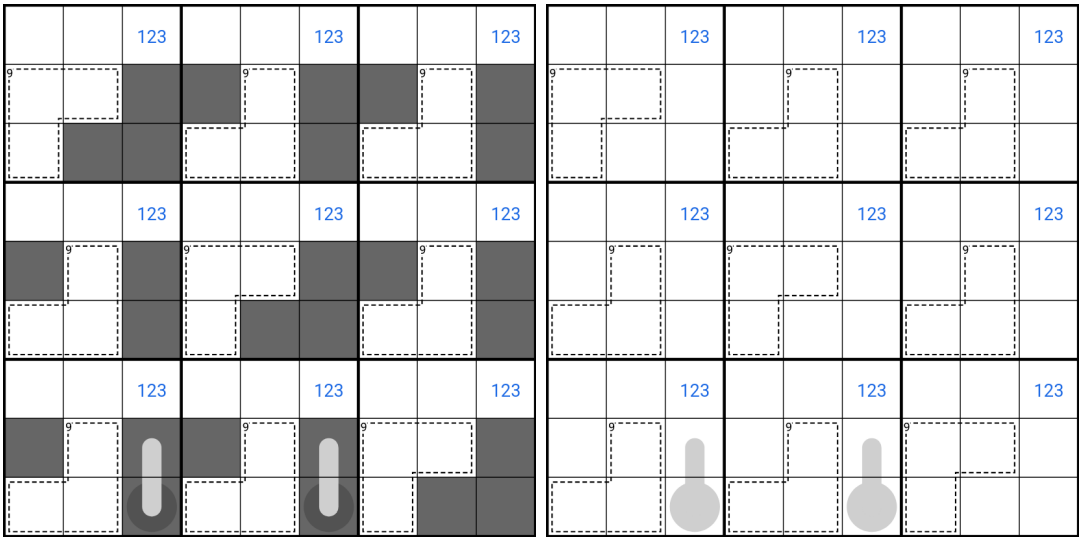
Let us take a look at the bottom two rows.



By observation 2, there are exactly 6 low digits inside the killer cages. Since there are 3 low digits in every row, this means that there are no low digits outside of the killer cages in these rows (indicated in black below). We can apply the same logic to rows 2 and 3, as well as to rows 5 and 6.



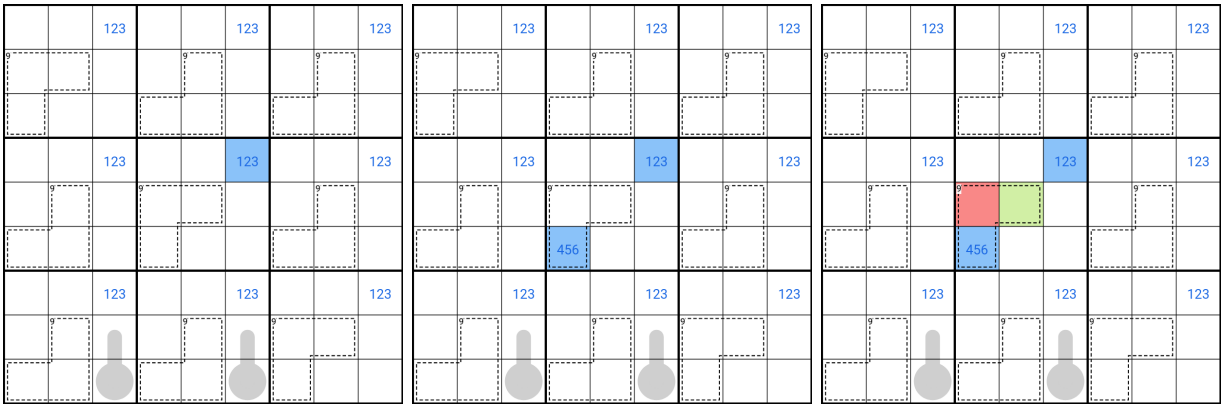
We can place the low digits in columns 3, 6, and 9.



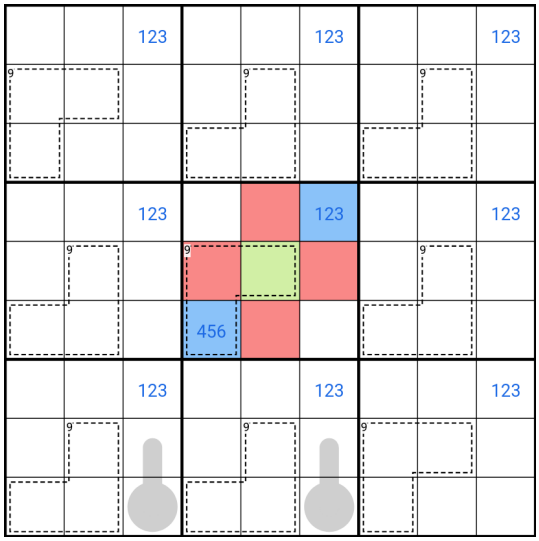
Now we will identify digits which are in the same modular class by colouring them in with the same colour. We will use the colours blue, green and red. Notice that with our colouring scheme, the final rule effectively says that we cannot colour two orthogonally adjacent digits the same colour.

If we focus on the middle box (though any box will do), we can colour the low digit in the top right corner blue (we do not yet know which modular class this digit belongs to, but we will figure this out later). Notice that the remaining two low digits are in the killer cage. Since these digits must be red and green, then in fact, the medium digit in the cage is also blue.

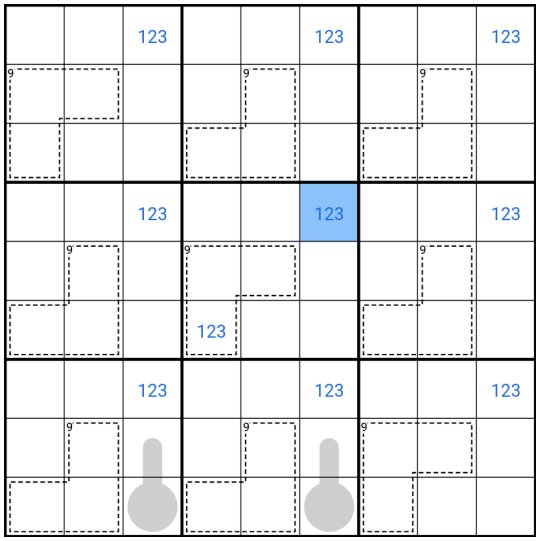
Suppose that the medium digit was in the bottom-left cell. Then we would have a situation like the following.



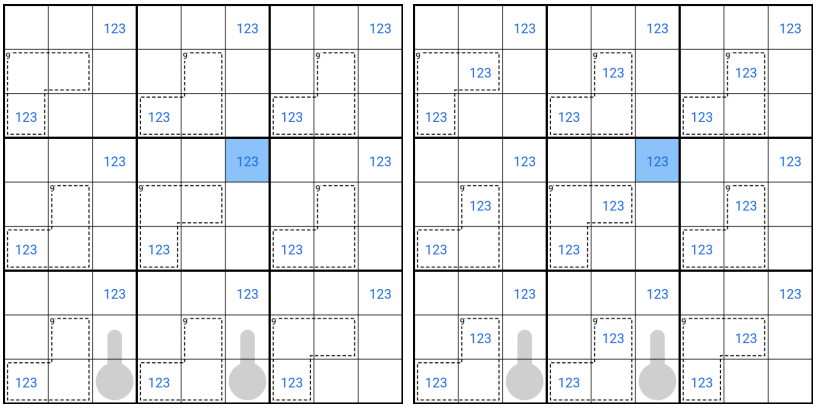
We have coloured the remaining low digits red and green (order does not matter). However, we have a bunch of other cells which border both a blue and the central green cell, so they all must be red, like so:



This gives 4 red cells in one box, which is impossible, so the medium blue cell is never in the bottom-left corner. This cell then must be one of the remaining low digits.

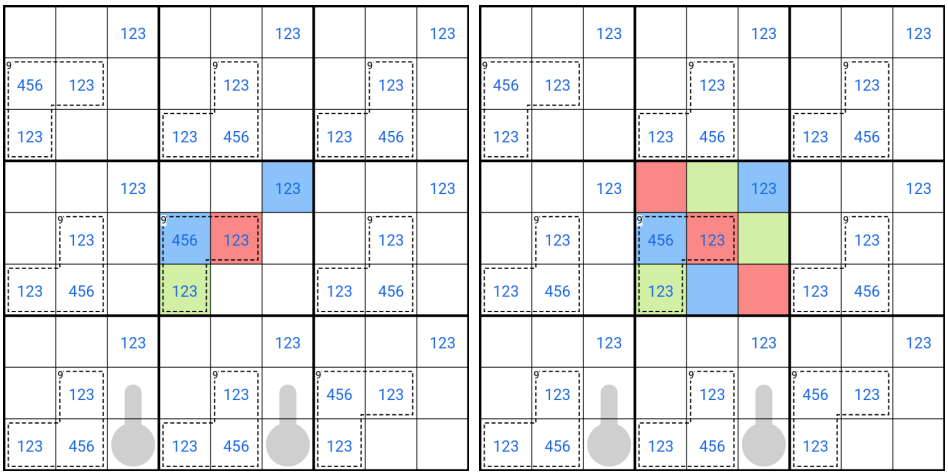


In fact, this is true for any box, so let us place low digits in all the remaining bottom-left cells.

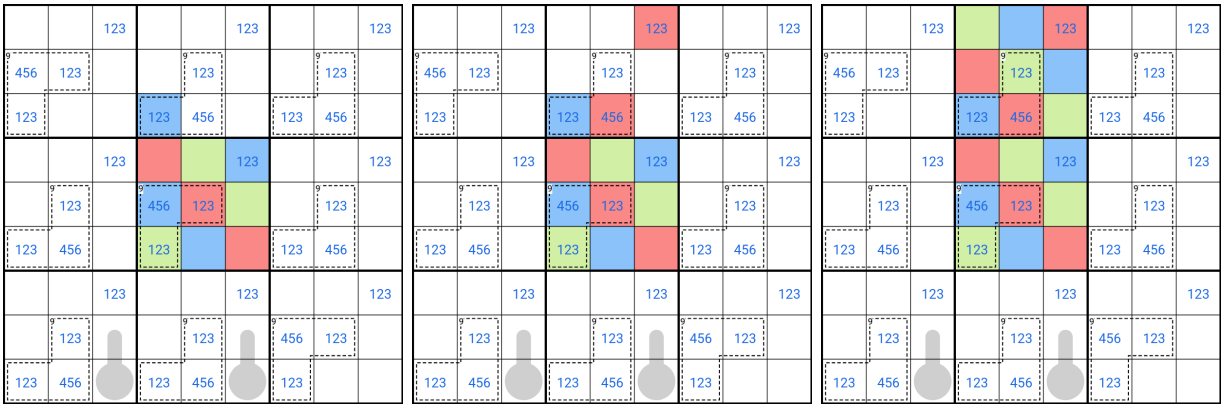


Some quick sudoku forces the remaining low digits to be placed in all the center cells. We can additionally place the medium digits in each of the cages.

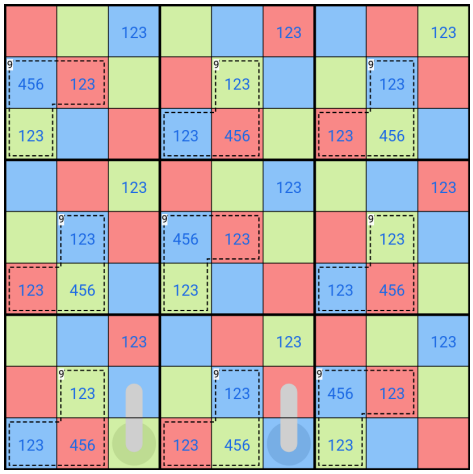
Now let us colour the digits in the central box. Of course, it does not matter which low digit we colour red or green (we can swap the colours)



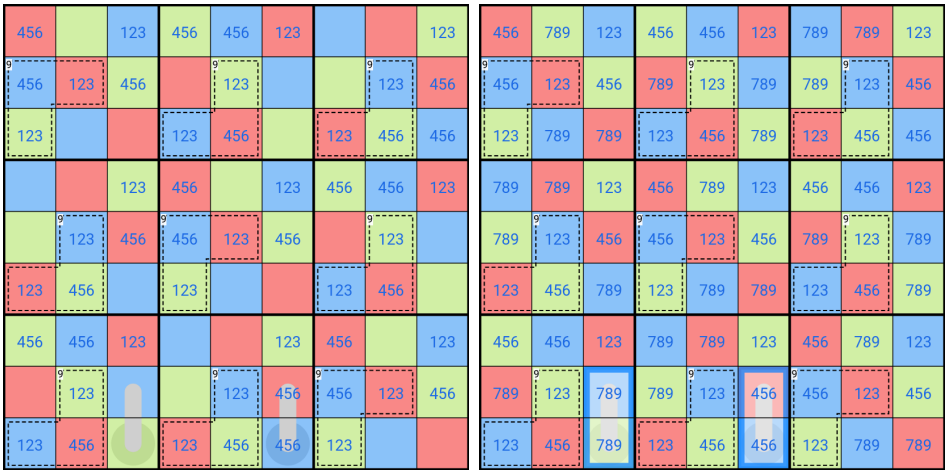
Again, all we've used is the fact that we cannot colour two neighbouring cells the same colour. We can also colour the bottom-left low digit in the box above. It is next to a red cell, so cannot be red, and it also sees the low green cell in the same column, so it too cannot be a (low) green cell. It therefore must be blue. We can colour it as such, and continue to colour the rest of the box.



Just as in the middle box, the medium cell in the cage must have the same colour as the low cell in the top-right corner. We can use this fact to help us colour the cage. Using this kind of rule, we can finish colouring the rest of the grid.

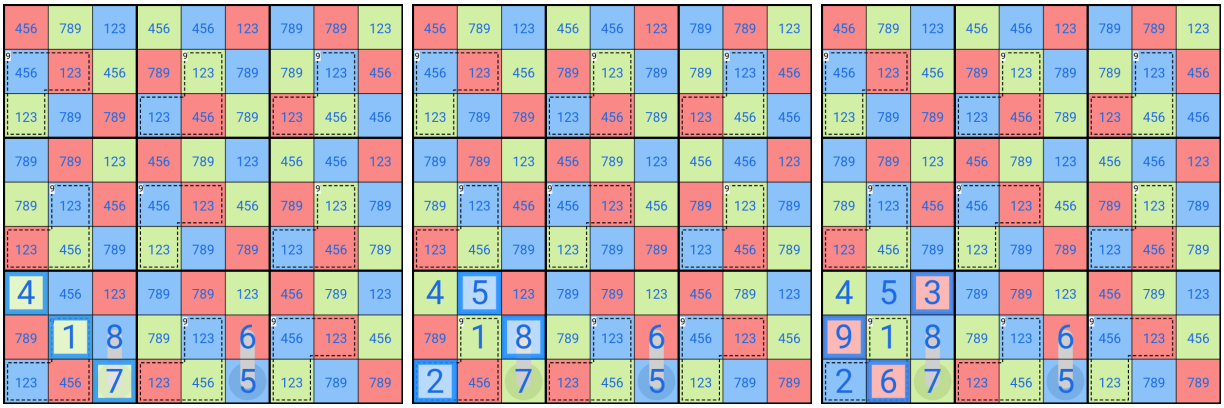


We can then use sudoku rules to place the remaining medium digits appropriately.

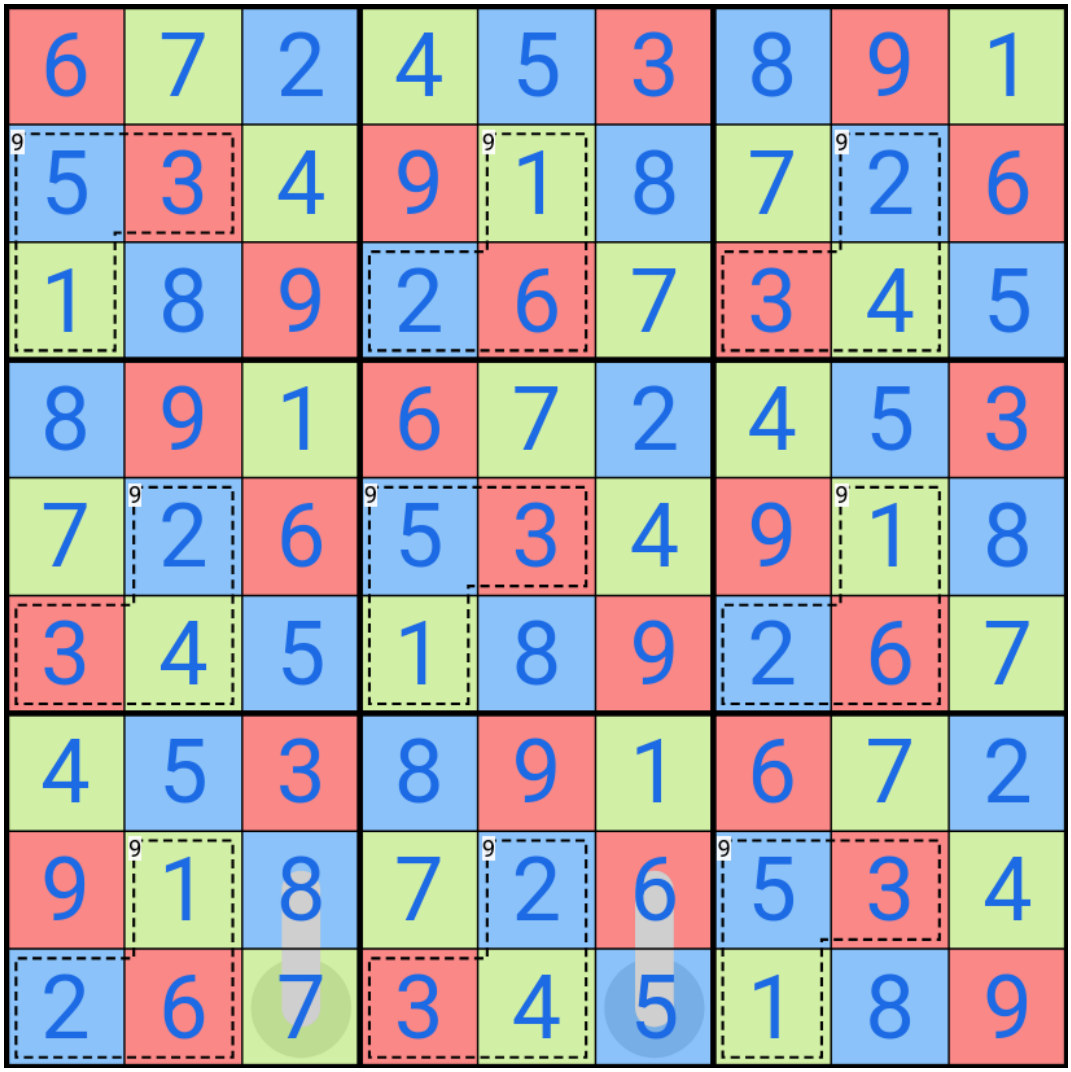


And the remaining digits must of course be high.

Finally, the two thermometers resolve the modular classes. We know that (within the low/medium/high digits), that green is less than blue, and blue less than red. So, green must be the digits $\{1, 4, 7\}$, blue must be $\{2, 5, 8\}$, and red $\{3, 6, 9\}$.



We can then place these digits, and suddenly the solution pops out! I've included it on the next page. Thanks for solving, I hope it was fun!



(The name "Nine Quilts" was inspired by the nine cages each whose sum was nine, as well as the final 'quilt-like' pattern that the puzzle produces at the end.)