Rajchman Algebras of Local Fields

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Fourier Analysis

- Start with a theorem:

Theorem

The Fourier transform is pretty cool.

- Classically, we can think of it as map $\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$ with

$$[\mathcal{F}(f)](s) = \int_{\mathbb{R}} f(x) e^{-2\pi i s x} \mathrm{d}x$$

- But the Fourier transform is not just a phenomenon on $\mathbb{R}$.
- We can also do it on a whole lot of other spaces.
- Specifically, we want locally compact abelian groups.

Pontryagin

- Let G be a LCAG.
- Then there's a dual group which we will call $\widehat{G}$, that **Pontryagin dual**.
- Some groups are self dual: $\widehat{\mathbb{R}} = \mathbb{R}$, $\widehat{C_n} = C_n$.
- Others are not, but the dual of a dual is always itself (so $\widehat{\widehat{G}} = G$). E.g. $\widehat{\mathbb{Z}} = \mathbb{T}$, $\widehat{\mathbb{T}} = \mathbb{Z}$.
- We can define a Fourier transform $\mathcal{F} : L^1(G) \to C_0(\widehat{G})$.
- This Fourier transform shares a lot of properties with the one we know and love.
    - E.g. $\mathcal{F}$ converts convolution operations to multiplication.

The Algebra twins

- We want to look at the range of $\widehat{\mathcal{F}} : L^1(\widehat{G}) \to C_0(G)$, which is a subalgebra of $C_0(G)$.
- We call this algebra $A(G)$, the **Fourier algebra** of G.
- There's actually a more powerful version of this transform, called the **Fourier-Stieltjes transform**.
- It maps $M(G) \to C_b(\widehat{G})$, and we call the range of this map the **Fourier-Stieltjes algebra**, denoted $B(G)$.
- So inside $C(G)$, there exist copies of both $L^1(\widehat{G})$ and $M(\widehat{G})$.
- Generally speaking $B(G)$ is big and trickier to describe, while $A(G)$ is more tractable.

An alternative way

- There is another way to think about these algebras $A(G)$ and $B(G)$.
- The secret is that $\widehat{G}$ is actually the collection of all irreducible representations (irreps) of G.
- The full computation is tricky, but the short result is as follows:
  - For an irrep π of G, we set

$$A_\pi(G) := \overline{\text{Span}}\{x \mapsto \langle \pi(x)\eta, \zeta \rangle : \eta, \zeta \in \mathcal{H}_\pi\}$$

  - Essentially, this is the (closed linear span of) functions which are 'matrix coefficients' of our irrep π.
  - Then $A(G) = A_\lambda(G)$, where λ is the left reg rep.
  - And

$$B(G) = \overline{\text{Span}} \bigcup_{\pi \in \widehat{G}} A_\pi(G)$$

Leaving Abelianland

- The previous formulation barely requires the need of an abelian group.
- If we let G be a (non-abelian) LCG, then we will set $\widehat{G}$ to be the collection of all irreps (not a group!)
- However, we can still define $A(G)$ and $B(G)$!
- It's as if we have an L^1 algebra and a measure algebra of a group that doesn't exist, hidden inside $C_b(G)$.
- These algebras can still us a lot about the underlying group G. (For instance $B(G) = A(G)$ if and only if G is compact).

P-adics crash course

- We now build up the p-adics.
- First we start with $\mathbb{Z}$, and we assign it an absolute value $|x|_p = p^{-n}$ where n is the largest power of p which divides x.
- This gives us a metric ($d(x, y) = |x - y|$), and completing $\mathbb{Z}$ with this metric gives us the ***p*-adic integers** $\mathbb{O}_p$.
- Key facts:
    - $\mathbb{O}_p$ is a PID, and it contains a sequence of nested ideals $p^n\mathbb{O}_p$ for $n \in \mathbb{N}$, with 0 being the unique element in all of them.
    - Topologically, it is a Cantor space (and thus compact).
    - The units $\mathbb{O}_p$ are precisely $\mathbb{O}_p^* = \mathbb{O}_p \setminus p\mathbb{O}_p$.

P-adics crash course part II

- Since $\mathbb{O}_p$ is an integral domain, we can extend it to its field of fractions.
- We call this field $\mathbb{Q}_p$, the *p*-**adic numbers**.
- (Alternatively, we can also construct $\mathbb{Q}_p$ by taking the completion of $\mathbb{Q}$ with respect to the metric on the previous slide.)
- This also has ideals $p^m\mathbb{O}_p$, but for $m \in \mathbb{Z}$.
- With the additive group structure, we have that $\mathbb{Q}_p$ is self-dual (i.e. $\widehat{\mathbb{Q}_p} = \mathbb{Q}_p$).

Semi-direct products and orbits

- Group of interest is $G = \mathbb{Q}_p^2 \rtimes \mathbb{O}_p^*$.
- Can think of as a matrix group

$$\begin{pmatrix} \mathbb{O}_p^* & \mathbb{Q}_p^2 \\ 0 & 1 \end{pmatrix}$$

- Primary question, what does $B(G)$ look like? We need to know the irreps!
- Turns out, the irreps depend on the orbits Z of the action $\mathbb{O}_p^* \curvearrowright \mathbb{Q}_p^2$. defined by $a \cdot (x, y) = (ax, ay)$.
- This action divides $\mathbb{Q}_p^2$ into lines (just like $\mathbb{R}^2$ for example), but also further divides into $\mathbb{Z}$ sections.
- Very roughly, Z will look like $\mathbb{Z}$ copies of $\mathbb{Q}_p$, and a trivial orbit at 0.

Dichotomy of irreps

- Looking further we find there are two types of irreps
- Those which 'live on' the $\mathbb{O}_p^*$ component. These are always finite dimensional.
- Those which 'live on' the $\mathbb{Q}_p^2$ component. These are always infinite dimensional, and correspond uniquely to some $z \in Z$ (with $z \neq 0$).
- For the latter, we denote these by π_z.
- We can break $B(G)$ into two parts:

$$B(G) = B_{PI}(G) \oplus B_{fin}(G)$$

  where $B_{PI}(G)$ are the functions which come from infinite irreps, and $B_{fin}(G)$ are those which come from finite reps.
- Since finite irreps 'live on' $\mathbb{O}_p^*$, we actually $B_{fin}(G) = A(\mathbb{O}_p^*)$.

Direct integrals

- Given a purely infinite rep (i.e. no finite subreps) π, we can decompose it as
$$\pi = \pi_\mu := \int_Z^{\oplus} \pi_z \mathrm{d}\mu(z)$$
- This notation needs some explaining!
- We have our orbit space Z, and each π_z is the irrep corresponding to the orbit $z \in Z$.
- μ is some positive measure on Z.
- The $\int^{\oplus}$ is a direct integral, essentially a continuous analogue of direct sums. Basically:
$$\frac{\int}{\sum} = \frac{\int^{\oplus}}{\bigoplus}$$

Measure structure

- If the measure μ is an atom (so $\pi_\mu = \pi_z$), then if $f \in A_{\pi_\mu}(G)$, we find that $f \in A(G/L)$ where L is some line in $\mathbb{Q}_p^2$.
- If μ is continuous (no atoms), we can show that $A_{\pi_\mu}(G)$ consists of functions which vanish at infinity.
- This gives us the following decomposition:

$$B_{PI}(G) = B_0(G) \oplus \left[\bigoplus_{L \text{ line}} A(G/L) \right]$$

where $B_0(G)$ are the elements of $B(G)$ which vanish at infinity (this is the **Rajchman algebra** of G).

- And so:

$$B(G) = B_0(G) \oplus \left[\bigoplus_{L \text{ line}} A(G/L) \right] \oplus A(\mathbb{O}_p^*)$$

How big is $B_0(G)$?

- Could it be possible that $B_0(G) = A(G)$?
  - True for the **Fell group** $\mathbb{Q}_p \rtimes \mathbb{O}_p^*$.
- We know that the left representation λ arises from a cts measure m on Z (called the *Plancherel* measure), so $A(G) \subseteq B_0(G)$.
- For any two measures μ, ν with $\mu \perp \nu$, we have that A_{π_μ} and A_{π_ν} are not contained in one another.
- Because Z contains a Cantor space, it turns out that it has an abundance of cts mutually singular measures.
- We can use these to find a cts measure μ such that $\mu \perp m$.
- So unfortunately $A(G) \neq B_0(G)$.

Thanks

Thanks for listening :)