Fourier Analysis of Local Fell Groups

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Overview

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Background: Local Fields

Background: Dual Spaces & The Fell Group

The Spine of Local Fell Groups

Algebra Amenability on Compact Fell-like Groups

Notation

- G: Locally compact, second-countable (Hausdorff) group.
- $\widehat{G}$: (Equivalence classes of) irreducible representations of G.
- $\mathbb{O}_p$ and $\mathbb{Q}_p$: Resp. p-adic integers and numbers.

The following will be introduced shortly:

- $\mathscr{K}$: A (non-Archimedean) local field.
- $\mathscr{U}$ and $\mathscr{V}$: Unit circle and local space of $\mathscr{K}$.
- G_F and G_f: The Fell group and a ‘compact version’ of it.

Background: Local Fields

Local Fields

Definition

A **local field** is a non-discrete locally compact field.

- Come in two varieties, either **Archimedean** or **non-Archimedean**.
- Only Archimedean local fields are $\mathbb{R}$ and $\mathbb{C}$.
- Non-Archimedean local fields are (finite extensions of) the p-adics $\mathbb{Q}_p$, and the Laurent series over a finite field $\mathbb{F}_{p^k}(x)$.
- We are primarily interested in non-Archimedean local fields.
- All results we state can be generalised to non-Archimedean local fields.

Key Facts: Local Fields

- All local fields are self-dual (in the Pontryagin sense).
- All local fields possess a *absolute value* $|\cdot|$.
- We define the **unit circle** of $\mathscr{K}$ is $\mathscr{U} := \{x \in \mathscr{K} : |x| = 1\}$.
- For $\mathbb{C}$, the unit circle is $\mathbb{T}$.
- For $\mathbb{Q}_p$, the unit circle is $\mathbb{O}_p^*$

Local Spaces

Definition (4.30)

A **local space** $\mathscr{V}$ is a finite dimensional vector space over a local field $\mathscr{K}$.

- A vector space over $\mathscr{K}$ is a local space iff it is locally compact.
- Can always equip $\mathscr{V}$ with a symmetric form $[\,\cdot\,,\,\cdot\,] : \mathscr{V} \times \mathscr{V} \to \mathscr{K}$.
  - Gives a notion of *orthogonality*, but not necessarily *positivity*.
  - Implement self-duality of $\mathscr{V}$.
- We let $\mathbf{H}(\mathscr{V})$ denote the **hyperplanes** ($n-1$ dim. subspaces) of $\mathscr{V}$.
- If $\varphi \in \widehat{\mathscr{V}}$, then $\ker(\varphi)$ contains a hyperplane (uses a rank-nullity result).

Background: Dual Spaces & The Fell Group

Dual Spaces

- The **dual space** $\widehat{G}$ of G is the collection of irreducible representations (irreps) on G, up to unitary equivalence.
- There is a natural topology on $\widehat{G}$.
- Much structural data of G can be extracted from $\widehat{G}$.
- When G is abelian, $\widehat{G}$ is a group under pointwise multiplication.
    - Called the *Pontryagin dual* of G.
- Otherwise not a group, but there is nonetheless interesting behaviour under tensor products of reps.

Fourier Algebras

Definition (Eymard 1964)

The **Fourier-Stieltjes algebra** of a group is defined to be

$$B(G) := \{\langle \pi(\cdot)\xi \mid \eta \rangle : \pi \in \mathrm{Rep}(G),\ \xi, \eta \in \mathscr{H}_\pi\}$$

and the **Fourier algebra** of a group is

$$A(G) := \{\langle \lambda(\cdot) f \mid g \rangle : f, g \in L^2(G)\}$$

- Both are commutative Banach algebras, with $A(G)$ an ideal of $B(G)$.
- For abelian groups, $B(G) \cong M(\widehat{G})$ and $A(G) \cong L^1(\widehat{G})$ via the Fourier(-Stieltjes) transform.
- Allow for a 'non-commutative' version of Fourier analysis.
- $A(G)$ is 'small' whereas $B(G)$ is 'large'.
- Coincide precisely when G is compact.

Fell Group

- Defined by $G_F = \mathbb{Q}_p \rtimes \mathbb{O}_p^*$, where $\mathbb{O}_p^* \curvearrowright \mathbb{Q}_p$ by multiplication.
- Dual space is countable, despite G_F being noncompact.
- Due to this unusual property, it is often a candidate for counterexamples to natural conjectures.
- It will model many (all) groups in this talk.

The Fell Group is Small

Theorem (Runde-Spronk 2004)

Let G be a locally compact group (LCG). Then $AM_{op}(B(G)) < 5$ iff G is compact.

- $AM_{op}(B(G_F)) = 5$.
- So G_F is a 'small' non-compact group.

Theorem (Baggett 1972, Walter 1974)

The Fourier-Stieltjes algebra of G_F is $B(G_F) = A(G_F) \oplus A(\mathbb{O}_p^*) \circ q$ where $q : G_F \to \mathbb{O}_p^*$ is the quotient map.

- So far, (essentially) the only known example of such a group.
- The above is also true for any **local Fell group** $\mathscr{K} \rtimes \mathscr{U}$.
- So $B(G_F)$ is composed of (sums of) Fourier algebras.

The Spine of Local Fell Groups

The Spine $A^*(G)$

Definition (Ilie-Spronk 2007)

The **spine** of the Fourier-Stieltjes Algebra of G is

$$A^*(G) := \overline{\mathrm{Span}}\{u : u \in A(H) \circ \eta, (H, \eta) \in \mathrm{Hom}(G, \cdot)\}$$

- Define *Fourier algebra associated to* η to be $A_\eta(G) := A(H) \circ \eta$.
- Always a closed subalgebra of $B(G)$.
- If $B(G) = A^*(G)$, we say that G is **spinal**.
- So G_F is spinal, and again, this is (essentially) the only known example.

Question

Are there other spinal groups? What about $\mathscr{V} \rtimes \mathscr{U}$*?*

Cheap Groups

- The action of $\mathscr{U} \curvearrowright \mathscr{V}$ is special.
- For $a \in \mathscr{U}$ and $\varphi \in \widehat{\mathscr{V}}$, then if $a \cdot \varphi = \varphi$, either $a = 1$ or $\varphi = \mathbf{1}$.
- Very similar to freeness for group actions on a Hausdorff space.

Definition (5.36)

Let $G = A \rtimes K$ where A abelian and K (infinite) compact. If $k \cdot \varphi = \varphi$ implies that either $k = 1$ or $\varphi = \mathbf{1}$ for all $k \in K$, $\varphi \in \widehat{A}$, we say G is (infinitely) **cheap**.

- We shall implicitly assume K is infinite going forward.

Mackey Machine

- Due to the *Mackey machine*, cheap groups have a nice dual structure.
- Irreps. come in two flavours:
  - Infinite dim. with $\pi = \mathrm{Ind}_A^G(\sigma)$ for $\sigma \in \widehat{A} \setminus \{\mathbf{1}\}$.
  - Finite dim. with $\pi = \rho \circ q$ for $\rho \in \widehat{K}$.
- As a consequence, $G^{ap} = K$ for cheap groups.
- A result of Runde-Spronk states that $B(G) = B_\infty(G) \oplus A(G^{ap}) \circ \eta_{ap}$, where $B_\infty(G)$ is the space of coefficient functions of *purely infinite* representations.

Corollary (Theorem 5.43)

Let $G = A \rtimes K$ be cheap. Then $B(G) = B_\infty(G) \oplus A(K) \circ q$.

Brief Aside: Dual Topology

- Can compute the dual topology of cheap groups.
- Dashed lines partition $\widehat{A}$ into orbit structure $\mathcal{O}_K(\widehat{A})$.
- Note that $\widehat{K}$ is discrete.
- We have that G is *type I* (i.e. $\widehat{G}$ is T_0).

Fourier Spaces

- Easy to show that $G = \mathscr{V} \rtimes \mathscr{U}$ is cheap, so that $B(G) = B_\infty(G) \oplus A(\mathscr{U}) \circ q$
- Our goal is to understand $B_\infty(G)$.

Definition (Arsac 1976)

For $\pi \in \mathrm{Rep}(G)$, the **Fourier space** associated to π is

$$A_\pi(G) := \overline{\mathrm{Span}}\{\langle \pi(\cdot)\xi \mid \eta\rangle : \xi, \eta \in \mathscr{H}_\pi\} \subseteq B(G)$$

- So if we can understand $A_\pi(G)$ for all purely infinite $\pi \in \mathrm{Rep}(G)$, then we can understand $B_\infty(G)$.

Hyperplane Kernels

- Recall that if $\varphi \in \widehat{\mathscr{V}}$, then $\ker(\varphi)$ contains a (unique) hyperplane.
- Follows that $\ker(\mathrm{Ind}_{\mathscr{V}}^{G}(\varphi))$ contains the same hyperplane.
- Let $\pi = \mathrm{Ind}_{\mathscr{V}}^{G}(\varphi)$ and H the above hyperplane.
- Since G/H is a local Fell group, we get the following:

Proposition (5.50)

If $q_H : G \to G/H$ is the quotient map, then $A_\pi(G) \subseteq A_{q_H}(G)$.

Proposition (5.51)

Let $G = \mathscr{V} \rtimes \mathscr{U}$. Then

$$\bigoplus_{\pi \in \widehat{G}_\infty} A_\pi(G) = \bigoplus_{H \in \mathbf{H}(\mathscr{V})} A_{q_H}(G) \subseteq B_\infty(G)$$

Continuous Sums

- This is not the complete picture, direct sums are not enough!
- Since G is type I, we can write any representation as a **direct integral** $\pi_\mu = \int_{\widehat{G}}^{\oplus} \rho \, \mathrm{d}\mu(\rho)$.
- Essentially a continuous analogue of direct sums (but very technical).
- In the previous slide, we solved for measures with atoms.
- Below is the result for continuous (atomless) measures.

Proposition (5.59)

Let $G = \mathscr{V} \rtimes \mathscr{U}$ with $\dim \mathscr{V} = 2$. If μ is a positive continuous σ-finite measure on Z, then $A_{\pi_\mu}(G) \subseteq B_0(G) := B(G) \cap C_0(G)$.

The Full Picture

Theorem (5.60)

Let $G = \mathscr{V} \rtimes \mathscr{U}$ with $\dim \mathscr{V} = 2$. Then

$$B(G) = \underbrace{\overbrace{B_0(G)}^{\text{cts. } \mu} \oplus \overbrace{\left[\bigoplus_{H \in \mathbf{H}(\mathscr{V})} A_{q_H}(G) \right]}^{\text{atomic } \mu}}_{\text{inf. dim. reps}} \oplus \underbrace{A(\mathscr{U}) \circ q}_{\text{fin. dim. reps}}$$

Moreover, if $u, v \in B_0(G)$, then $uv \in A(G)$.

The Full Picture 2: Electric Boogaloo

Theorem (5.61)

Let G be a LCG. Then $B_0(G) \cap A^*(G) = A(G)$.

Corollary (5.62)

Let $G = \mathscr{V} \rtimes \mathscr{U}$ with $\dim \mathscr{V} = 2$. Then

$$A^*(G) = \underbrace{\overbrace{A(G)}^{\text{cts. } \mu} \oplus \overbrace{\left[\bigoplus_{H \in \mathbf{H}(\mathscr{V})} A_{q_H}(G) \right]}^{\text{atomic } \mu}}_{\text{inf. dim. reps}} \oplus \underbrace{A(\mathscr{U}) \circ q}_{\text{fin. dim. reps}}$$

Failure of Spinality

Question

For $G = \mathscr{V} \rtimes \mathscr{U}$ with $\dim(\mathscr{V}) = 2$, is it possible that $B_0(G) = A(G)$?

- Unfortunately no!
- Note that $\widehat{G}$ contains a Cantor space, which has an abundance of continuous mutually singular measures.
- All continuous measures have $A_{\pi_\mu}(G) \subseteq B_0(G)$.
- But we can also find one such that $A_{\pi_\mu}(G)$ is disjoint from $A(G)$.
- So $A(G) \neq B_0(G)$, and so G is not spinal.

Algebra Amenability on Compact Fell-like Groups

Brief History of (Banach Algebra) Amenability

Theorem (Johnson 1972)

G is amenable if and only if $L^1(G)$ is amenable.

Theorem (Forrest, Runde 2004)

G is virtually abelian if and only if $A(G)$ is amenable.

Henceforth we assume G is compact.

Conjecture (Azimifard, Samei, Spronk 2009)

G is virtually abelian if and only if $ZL^1(G)$ is amenable.

- Forward direction shown by Alaghmandan and Crann in 2017.

Central Fourier Algebra

Definition (Alaghmandan, Spronk 2017)

The **central Fourier algebra** of G is

$$ZA(G) := A(G) \cap ZL^1(G) = \{u \in A(G) : u(xy) = u(yx), \ \forall x, y \in G\}$$

Theorem (Alaghmandan, Spronk 2017)

If G is virtually abelian then $ZA(G)$ is amenable.

- Converse direction is conjectured true.
- Consider $G_f := \mathbb{O}_p \rtimes \mathbb{O}_p^*$, the compact sister of the Fell group.
- G_f is not virtually abelian, so is $ZA(G_f)$ amenable?

K-Central Fourier Algebra

Definition

For $K \curvearrowright G$, we set the K**-central Fourier algebra**

$$Z_K A(G) := \{u \in A(G) : u(k \cdot x) = u(x),\ \forall x \in G, k \in K\}$$

- When G acts on itself by conjugation, then $Z_G A(G) = ZA(G)$.

Proposition (6.6)

If G abelian (and K compact) with $K \curvearrowright G$, then $Z_K A(G)$ is a quotient algebra of $ZA(G \rtimes K)$.

- This algebra $Z_K A(G)$ is the Fourier algebra of a *hypergroup*.

Hypergroups

- A hypergroup H is a generalisation of a group.
- Given by a convolution $* : M(H) \times M(H) \to M(H)$ with some properties.
- Many usual group constructions work similarly.
- The orbits of $K \curvearrowright G$ form a hypergroup.

Theorem (Muruganandam 2008)

Let K be compact acting on G, with associated hypergroup H. Then $Z_K A(G) \cong A(H)$.

Theorem (6.18)

Let K be compact acting on an abelian G. Let H be the hypergroup of $K \curvearrowright G$, and $\widehat{H}$ for $K \curvearrowright \widehat{G}$. Then $L^1(H) \cong A(\widehat{H})$.

Application

- We can compute the dual hypergroup $\widehat{H}$ of G_f explicitly.
- Satisfies some amenability-like property called (P_2), and we can use the following result.

Theorem (Alaghmandan 20)

Let H be a discrete commutative hypergroup satisfying (P_2). If $\{x \in H : |x| \leq M\}$ is finite for every $M \geq 0$, then $L^1(H)$ is not amenable.

- $\widehat{H}$ also has that $\{x \in \widehat{H} : |x| \leq M\}$ is finite, so $L^1(\widehat{H})$ is not amenable.
- Thus $Z_{\mathbb{O}_p^*}A(\mathbb{O}_p) \cong A(H)$ is not amenable.
- So $ZA(G_f)$ is not amenable, reaffirming the conjecture.

Thank you

Appendices

Fell Group Dual is *Countable*

- In my thesis, I state that G_F is a non-discrete group whose dual is compact.
- However, this is **not true**!
- The dual of G_F is not compact, though it is in a sense 'close' to being compact.
- In a similar manner, it is also a noncompact group whose dual is 'almost' discrete.
- What is actually true however, is that it is a noncompact group whose dual is *countable*.
- This could never occur in the abelian case, as all countable groups are always discrete.

Baggett's Result

- Baggett's paper shows that a group is compact if and only if its dual is discrete.
- The 'converse' to this is still not true: namely there is a non-discrete group whose dual is compact.
- Also identified by Baggett.
- This group is $G = \mathbb{R} \rtimes \mathbb{Z}$ where $\mathbb{Z} \curvearrowright \mathbb{R}$ by $n \cdot r = e^n r$.
- Both the orbit space and $\widehat{\mathbb{Z}} \cong \mathbb{T}$ are compact in this case, which suggests that $\widehat{G}$ should be too (as is the case).