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The Picard Theorems and the Riemann Mapping Theorem

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Abstract

We look at various interesting results from complex analysis. In particular we focus on the Picard Theorems and the Riemann Mapping Theorem. The former of these tells us that image of a holomorphic function omits at most one value under certain conditions. The Riemann Mapping Theorem gives us an equivalence relation between all proper simply connected subsets of the complex plane. We prove these as well as other interesting auxiliary results.

Contents

Introduction

In this report, we shall explore many different theorems in complex analysis. Our particular focus will be in proving the Picard Theorems and the Riemann Mapping Theorem. In order to prove these, we will end up proving and using several other results along the way.

The Picard Theorems state that the image of a holomorphic function can omit at most one value under certain conditions. For the Little Theorem, this condition is that f is non-constant and holomorphic on all of $\mathbb{C}$. For the Great Theorem, the condition is that f is holomorphic in a punctured neighbourhood of an essential singularity.

The Riemann Mapping Theorem states that all proper simply connected subsets of the complex plane are biholomorphic to the unit disk, and that moreover this mapping is 'essentially unique'. Here biholomorphic means that our mapping is holomorphic, and its inverse is holomorphic as well.

We shall assume the reader is familiar with some basic complex analysis definitions and tools. Being familiar with the MATH312 course would be useful, though not necessarily required. In particular, we assume that the reader knows the definition of a holomorphic function, and knows several theorems pertaining to these. To name a few, we shall frequently use Cauchy's Theorem and Integral Formula, as well as Morera's Theorem. Familiarity with topological concepts such as compactness and connectedness shall also be necessary.

Our approach in the proofs of these shall follow loosely what is set out in chapters 8 and 10 of Remmert [3]. We do use other chapters of Remmert as well portions of Conway [2] in proving several auxiliary results along the way.

Common Definitions

We use the following definitions throughout this report.

The set $\mathbb{N}$ denotes the natural number specifically without the element 0. The sets $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ have their usual meaning.

We define **disks** as

$$D(p;r) := \{z \in \mathbb{C} : |z - c| < r\}$$

and similarly define **closed disks** as

$$\overline{D}(p;r) := \{z \in \mathbb{C} : |z - c| \leq r\}$$

Of particular interest is the **unit disk**, which is defined and denoted as

$$\mathbb{D} := D(0;1)$$

and the similarly the closed unit disk

$$\overline{\mathbb{D}} = \overline{D}(0;1)$$

We shall also define a **circle** in $\mathbb{C}$ to be the boundary of a disk oriented anticlockwise. In other words

$$C(p;r)(t) := re^{it} + p$$

with $t \in [0, 2\pi]$.

It is also assumed the reader is familiar with the notion of a winding number, but for completeness this is defined as follows

$$n(\gamma; z_0) := \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - z_0}$$

Chapter 1

Picard's Little Theorem

In this first section, we aim to prove Picard's Little Theorem. This theorem tells us that any non-constant holomorphic function on $\mathbb{C}$ can omit at most one value from its image. To this end we first build up a few more tools that we shall use throughout this text. Of note, we prove Rouché's Theorem and the open Mapping Theorem.

We shall then look at Bloch's Theorem, which gives a lower bound on the size of disks in the image of a holomorphic function. This theorem is a major stepping stone towards proving Picard's Little Theorem - after Bloch, we need only a few more lemmas to prove this.

1.1 Counting Zeroes and the Open Mapping Theorem

In MATH312, we previously only defined a function to be holomorphic on an open set. We can in fact extend this to any arbitrary set as follows.

Definition 1.1.1. *We say that a function f is **holomorphic** on an arbitrary set E if f is holomorphic on some open neighbourhood of E.*

Our first focus is counting zeroes of any given holomorphic function, which will eventually lead us to Rouché and from that the open mapping Theorem. In this first lemma, we show that essentially the number of zeroes of a holomorphic function is finite (or else converges outside of the domain of f).

For this lemma we shall use the fact that if f has a sequence of zeroes converging to some point w, then f is identically zero on some disk about w.

Lemma 1.1.2. *Let f be a non-zero holomorphic function on a region U. Then the set of zeroes of f in U does not have a limit point in U.*

Proof. Let X be the set of zeroes of f, and suppose it has a limit point. Define the set

$$V = \{z \in U : \text{There is a neighbourhood } W \text{ of } z \text{ such that } f(u) = 0 \text{ for every } u \in W\}.$$

We can see clearly from the definition that V is open as it can be expressed union of neighbourhoods.

Now take any $w \in \partial V$. Let $D_n = D(w; 1/n) \cap V$. Since D_n is an open neighbourhood of w, and w is on the boundary of V then, there is some $w_n \in D_n \cap V$. From this definition, w_n

converges to w. Moreover $f(w_n) = 0$. This means that f is zero on some disk about w. But this means that $w \in V$. So V contains its boundary, and is therefore closed.

Lastly, X has a limit point, call it a. Then we know that f must be zero on some disk about a. So $a \in V$, and so V is non-empty.

So V is a clopen set in U, and is non-empty. Since U is connected, $V = U$ (otherwise it would be a separation of U). So f is identically zero on U. □

Now, this allows to already get to a significant theorem. This theorem (the argument principle) will allow us to count the number of zeroes of a function f in any region of the plane. This will turn out to be incredibly useful, as if we take any $c \in \mathbb{C}$, and look at the function $f - c$, we see that this allows us to count the number of time f attains the value c in its image.

Theorem 1.1.3 (Argument Principle). *Let f be holomorphic on a region U. Let γ be a closed curve homologous to 0 in U so that $0 \notin f(\gamma)$. Let X be the set of zeroes of f in U. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{w \in X} m(w) n(\gamma; w) \tag{1.1}$$

where $m(z_0)$ is the multiplicity of z_0.

Proof. Firstly, since $0 \notin f(\gamma)$, then f is not identically zero on U. So by Lemma 1.1.2, then X has no limit point in U.

Now let $E = \{z \in \mathbb{C} \setminus \gamma : n(\gamma; z) \neq 0\}$. Since γ is homologous to 0, it follows that $E \subseteq U$. Now, we know that $n(\gamma; z)$ is constant on each of the connected subsets of $\mathbb{C} \setminus \gamma$. In particular if S is an unbounded connected subset, then $n(\gamma; z) = 0$ for $z \in S$. This means that E is a bounded set. Moreover $\overline{E}$ is also bounded, and therefore compact.

The boundary of E will contain elements which are arbitrarily close to points z_1 and z_2 with $n(\gamma; z_1) = 0$ and $n(\gamma, z_2) \neq 0$. However this can only happen on γ itself. So this means that $\overline{E} \subseteq E \cup \gamma \subseteq U$.

Now we need only consider X as a subset of E - everywhere else $n(\gamma; z) = 0$ and therefore does not contribute to the sum. So if X is infinite, then it is infinite in E, and will therefore have a limit in $\overline{E}$ and hence a limit in U. We have already seen this is cannot occur, so X must be finite.

Since X is finite we can write f as

$$f(z) = \left(\prod_{w \in X} (z - w)^{m(w)} \right) h(z)$$

for some holomorphic function h. Note that h will have no zeroes in E, so that $h'(z)/h(z)$ is holomorphic with

$$\int_\gamma \frac{h'(z)}{h(z)}\, \mathrm{d}z = 0$$

Now, as X is finite we can use a repeated product rule to see that

$$f'(z) = f(z) \left[\sum_{w \in X} \frac{m(w)}{z - w} + \frac{h'(z)}{h(z)} \right]$$

Now evaluating the integral we get that

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\ \mathrm{d}z &= \frac{1}{2\pi i}\int_\gamma \left[\sum_{w\in X}\frac{m(w)}{z-w}+\frac{h'(z)}{h(z)}\right]\ \mathrm{d}z \\ &= \frac{1}{2\pi i}\left[\sum_{w\in X}\int_\gamma \frac{m(w)}{z-w}\ \mathrm{d}z+\int_\gamma\frac{h'(z)}{h(z)}\right]\ \mathrm{d}z \\ &= \sum_{w\in X} m(w)\frac{1}{2\pi i}\int_\gamma \frac{1}{z-w}\ \mathrm{d}z \\ &= \sum_{w\in X} m(w)n(\gamma;w)\end{aligned}$$

□

In fact, as an interesting aside, the formula in eq. (1.1) can be extended to include poles as well, by writing

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\ \mathrm{d}z = \sum_{w\in X_z} m_z(w)n(\gamma;w) - \sum_{v\in X_p} m_p(v)n(\gamma;v)$$

We will not need this so the proof is omitted, but one may see that the argument for is almost identical to the above.

We can now prove Rouchés Theorem, which gives us an alternative to counting zeroes using the argument principle. In particular, it allows us to get the number of zeroes by comparing to a function whose zeroes we already know. This is very useful, and we will use it (directly or indirectly) in proving other results in this text. (For example Hurwitz's Injection Theorem - see Lemma 2.2.1).

Theorem 1.1.4 (Rouché's Theorem)**.** *Let f and g be holomorphic on some region U, which has a closed curve γ as its boundary. Suppose for all $z \in U$, we have that $n(\gamma, z) = 1$. Further, suppose that for all $z \in \gamma$, we have that*

$$|f(z) - g(z)| < |f(z)| + |g(z)| \tag{1.2}$$

Then f and g have the same number of zeroes in D, counting multiplicity.

Proof. Firstly, define $h(z) := f(z)/g(z)$. Note that by condition (1.2), we have that g is non-zero on the boundary, so h is defined at all points along γ. Moreover, if we divide through by $|g(z)|$, we get that $|h(z) - 1| < |h(z)| + 1$.

Let σ be defined by the closed curve $h \circ \gamma$. Then we have that

$$\begin{aligned}\int_\gamma \frac{h'(z)}{h(z)}\ \mathrm{d}z &= \int_0^1 \frac{h'(\gamma(t))}{h(\gamma(t))}\gamma'(t)\ \mathrm{d}t \\ &= \int_0^1 \frac{[h(\gamma(t))]'}{h(\gamma(t))}\ \mathrm{d}t \\ &= \int_0^1 \frac{\sigma'(t)}{\sigma(t)}\ \mathrm{d}t \\ &= \int_\sigma \frac{1}{z}\ \mathrm{d}z\end{aligned}$$

Now, suppose that $w \in \mathbb{C}$ such that w satisfies

$$|w - 1| < |w| + 1 \tag{1.3}$$

Let $w = x + iy$. Then we get the following result about w.

$$\begin{aligned} |w-1| &< |w| + 1 \\ |(x-1) + iy| &< |x + iy| + 1 \\ \sqrt{(x-1)^2 + y^2} &< \sqrt{x^2+y^2} + 1 \\ x^2 + 1 - 2x + y^2 &< x^2 + y^2 + 1 + 2\sqrt{x^2+y^2} \\ -x &< \sqrt{x^2+y^2} \end{aligned}$$

We can see that the above fails precisely when $x \leq 0$ and $y = 0$ (i.e. it fails for non-positive real numbers). Note this forms a simply connected region, and in particular, a region which the function $1/z$ is holomorphic on.

Now σ is a curve such that for all t, we have that $\sigma(t)$ satisfies equation (1.3). This means that by Cauchy's Theorem we get that

$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}\,\mathrm{d}z = \frac{1}{2\pi i}\int_\sigma \frac{1}{z}\,\mathrm{d}z = 0$$

Now from our definition of h, we can see that

$$\begin{aligned} \frac{h'(z)}{h(z)} &= \frac{[f(z)/g(z)]'}{f(z)/g(z)} \\ &= \frac{g(z)}{f(z)}\frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2} \\ &= \frac{f'(z)g(z) - f(z)g'(z)}{f(z)g(z)} \\ &= \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)} \end{aligned}$$

If we let N_f and N_g denote the number of zeroes of f and g respectively, then we have that

$$\begin{aligned} N_f - N_g &= \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z - \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}\,\mathrm{d}z \\ &= 0 \end{aligned}$$

Hence f and g have the same number of zeroes. □

This leads us the following Corollary, which will use in our proof of the open mapping theorem, as well as proving Bloch's Theorem.

Corollary 1.1.5. *Let f be holomorphic on a closed disk $\overline{D} = \overline{D}(0; r)$. Suppose that f has at least one zero in $\overline{D}$, and that for every $z \in \partial\overline{D}$, we have $R \leq |f(z)|$ for some $R > 0$. Then we have that $D(0; R) \subseteq f(\overline{D})$.*

Proof. Let $w_0 \in D(0; R)$, so that we have that $|w_0| < R$. Define $g(z) = f(z) - w_0$. It then follows that

$$|f(z) - g(z)| = |w_0| < R \leq |f(z)| \leq |f(z)| + |g(z)|$$

for every $z \in \overline{D}$.

By Theorem 1.1.4, we then have that g must have at least one zero. So there is some $z_0 \in D(0;R)$ such that $f(z_0) = w_0$. This is true for all $|w| < R$, hence $D(0;R) \subseteq f(\overline{D})$. $\square$

We now have enough to prove the open mapping theorem. This theorem tells us that any non-constant holomorphic function is an open map - in other words, the image of any open set is also open.

Theorem 1.1.6 (Open Mapping Theorem)**.** *Let f be a holomorphic function on some region U. If f is non-constant, then f is an open map.*

Proof. Take an open $V \subseteq U$. To show that f is an open map, we need to show that every point in $f(V)$ is inside an open disk contained in $f(V)$. So take $w_0 \in f(V)$, and let $z_0 \in V$ be such that $f(z_0) = w_0$. Moreover, since V is open, we must have some $r > 0$ such that $D(z_0;r) \subseteq V \subseteq U$.

Now, define $g(z) = f(z + z_0) - w_0$. We immediately have that f is holomorphic on $D(0;r)$ and contains a zero in this disk. Next we want to find a disk whose boundary does not contain a zero (of g). We can find an r' with $0 < r' < r$, such that $\partial D(0;r')$ contains no zeroes. If we could not, then $\overline{D(0;r/2)}$ would necessarily contain infinitely many zeroes. However this would mean that g would be zero everywhere on U, and so f would be constant.

Now let $C = \partial D(0;r')$. Since C is compact, $|g(z)|$ must achieve its infimum. Since g has no zeroes on C, it then follows that there is some $R > 0$ such that $|g(z)| > R$ for every $z \in C$. We can now apply Corollary 1.1.5 to g, so we have that

$$D(0;R) \subseteq g(D(0;r'))$$

Rewriting this in terms of f:

$$D(w_0;R) \subseteq f(D(z_0;r')) \subseteq f(D(z_0;r)) \subseteq f(V)$$

Hence f is open. $\square$

This theorem is very powerful, and we will also use this to prove several theorems throughout this text. Of note, is how different this theorem is from the real case. A very simple counterexample is $x \to x^2$, where the image of $(-1,1)$ is $[0,1)$ which is not open (in fact any open set containing 0 will achieve this).

The function $x \to x^2$ is far from the only counterexample - any function with a local maximum or minimum will not be an open map (in the reals). In other words, any real open map will not any local extrema. One might expect something similar to be true for the complex case (where every non-constant holomorphic map is open). And indeed we do, as we have below.

Theorem 1.1.7 (Maximum Principle)**.** *Let f be holomorphic on a region U and let $z_0 \in U$. Suppose that $|f(z_0)| \geq |f(z)|$ for every $z \in U$. Then f is constant.*

Proof. Suppose toward contradiction that f is non-constant. It follows by the open mapping theorem (Theorem 1.1.6) that f is an open map. If we let $w_0 = f(z_0)$, it follows that $f(U)$

contains an open neighbourhood of w_0. This means that there is some r such that the disk $D(w_0; r) \subseteq f(U)$. If we define

$$w := w_0 \left(1 + \frac{r}{2|w_0|}\right)$$

it is not difficult to see that $|w| > |w_0|$ and that $w \in D(w_0; r)$. However this means that $w \in f(U)$, so $w = f(z)$ for some $z \in U$, so $|f(z)| > |f(z_0)|$. This is a contradiction, so f must be constant. $\square$

From this we obtain also the following corollary.

Corollary 1.1.8. *Let f be holomorphic on an open neighbourhood of some connected compact set F. Suppose there is some $M \in \mathbb{R}$ such that $|f(z)| \leq M$ for all $z \in \partial F$. Then $|f(z)| \leq M$ for all $z \in F$.*

Proof. This is trivial in the case that f is constant. So suppose f is non-constant. As F is compact and the mapping $z \mapsto |f(z)|$ is continuous, then this map must attain its maximum at some $w \in F$. However as f is non-constant, then by the maximum principle, w cannot be in the interior of F. So $w \in \partial F$. Hence for every $z \in F$, $|f(z)| \leq |f(w)| \leq M$. $\square$

While these two statements are interesting in their own right, we shall not need them just yet. They will be useful later in the proof of Picard's Great Theorem.

1.2 Bloch's Theorem

We next focus on arguably the most important step in our proof of Picard's Little Theorem. This step is Bloch's Theorem, which gives us (rather surprisingly) a lower bound on the size of disks in the image of a holomorphic function. The exact formulation of the statement of the theorem is given below.

Theorem 1.2.1 (Bloch's Theorem). *If f is holomorphic on the closed unit disk $\overline{\mathbb{D}}$, and $f'(0) = 1$, then the image of f contains a disk of radius $\frac{3}{2} - \sqrt{2}$.*

This statement is again in quite a sharp contrast to the real situation. In the real case, one can relatively easily see that it is possible to make the image of such functions as small as possible, even by keeping the derivative constant.

The number $\frac{3}{2} - \sqrt{2} \approx 0.0858$ here is not anything special - it is merely a (non-zero) lower bound to the size of such disks. Specifically if we modify the statement of theorem to say the f contains a disk of radius B, then the largest such B is known as Bloch's constant, and the precise value of B is currently not known. It has however been shown (see [1]) that B is at least $\frac{\sqrt{3}}{4} + 2 \times 10^{-4} \approx 0.4332$.

For our case though, any non-zero bound will suffice. While we obtain the bound $\frac{3}{2} - \sqrt{2}$ in our proof, we shall later simply reduce this to $\frac{1}{12}$ (which is smaller than $\frac{3}{2} - \sqrt{2}$) for the sake of brevity.

The main lemma in the proof of Bloch's Theorem is as follows. Here, $|f|_D$ denotes the supremum of $|f|$ on D.

Lemma 1.2.2. *Let f be holomorphic on $D = D(a; r)$ and non-constant, with the additional property that $|f'|_D \leq 2|f'(a)|$. Then*

$$D(f(a); R) \subseteq f(D), \qquad \text{where } R = (3 - 2\sqrt{2})r|f'(a)| \tag{1.4}$$

Proof. Firstly, we can assume that $a = f(a) = 0$. To see why this is the case, let $g := f(x+a) - f(a)$ and $D' = D(0;r)$. It follows that g is holomorphic on D', non-constant and that $|g'|_{D'} \leq 2|g'(0)|$ (as $g'(x) = f'(x+a)$). Moreover, we have that $g(0) = 0$. In other words we have reduced the problem to the case where $a = f(a) = 0$.

So, suppose we have $a = f(a) = 0$. Define a function $A(z) = f(z) - f'(0)z$. It follows that

$$A(z) = \int_{\overrightarrow{[0,z]}} (f'(w) - f'(0)) \, \mathrm{d}w$$

Using the substitution $w = zt$, we find that

$$|A(z)| \leq \int_0^1 |f'(zt) - f'(0)||z| \, \mathrm{d}t$$

Now take any $z \in D$ and let $\varepsilon > 0$ so that $z \in D(0; r - \varepsilon)$. Let γ be the curve going around the circle $C(0; r - \varepsilon)$ once anticlockwise. Now, using Cauchy's Integral formula, we have that

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f'(w)}{w - z} \, \mathrm{d}w$$

Hence

$$\begin{aligned} f'(z) - f'(0) &= \frac{1}{2\pi i} \int_\gamma \left(\frac{f'(w)}{w - z} - \frac{f'(w)}{w} \right) \mathrm{d}w \\ &= \frac{z}{2\pi i} \int_\gamma \frac{f'(w)}{w(w-z)} \, \mathrm{d}w \end{aligned}$$

Now the length of the curve γ is $2\pi(r - \varepsilon)$, therefore

$$\begin{aligned} |f'(z) - f'(0)| &\leq \frac{|z|}{2\pi} \int_\gamma \frac{|f'(w)|}{|w||w-z|} \, \mathrm{d}w \\ &\leq \frac{|z|}{2\pi} \frac{|f'|_D}{(r-\varepsilon)(r - \varepsilon - |z|)} 2\pi(r - \varepsilon) \\ &= \frac{|z|}{(r - \varepsilon - |z|)} |f'|_D \end{aligned}$$

Since this is true for every $\varepsilon > 0$, we must have that

$$|f'(z) - f'(0)| \leq \frac{|z|}{(r - |z|)} |f'|_D$$

We can then use this to get a bound for $A(z)$ as follows:

$$\begin{aligned} |A(z)| &\leq \int_0^1 |f'(zt) - f'(0)||z| \, \mathrm{d}t \\ &\leq \int_0^1 \frac{|zt|}{(r - |zt|)} |f'|_D |z| \, \mathrm{d}t \\ &\leq \int_0^1 \frac{|zt|}{(r - |z|)} |f'|_D |z| \, \mathrm{d}t \\ &= \frac{1}{2} \frac{|z|^2}{(r - |z|)} |f'|_D \end{aligned}$$

Expanding the definition of A, and using our assumption that $|f'|_D \leq 2|f'(0)|$, we get that

$$\begin{aligned}
|f'(0)||z| - |f(z)| \leq |f(z) - f'(0)z| &= |A(z)| \\
&\leq \frac{1}{2}\frac{|z|^2}{(r-|z|)}|f'|_D \\
&\leq \frac{|z|^2}{(r-|z|)}|f'(0)| \\
\left(|z| - \frac{|z|^2}{r-|z|}\right)|f'(0)| \leq |f(z)|
\end{aligned}$$

Now let $\rho = |z|$. Note that ρ can take values precisely in the range $(0, r)$. Using some basic calculus, we can see that the expression $\rho - \rho/(r-\rho)$ achieves its maximum value of $(3-2\sqrt{2})r$ at $\rho = (1-\sqrt{2}/2)r$.

$$R = (3-2\sqrt{2})r|f'(0)| \leq |f(z)|$$

If we let $B = D(0; R)$, then by Corollary 1.1.5, we get that $B \subseteq f(B)$. □

This lemma is enough to let us prove Bloch's Theorem

Proof of Bloch's Theorem. Let f be holomorphic on $\overline{\mathbb{D}}$ and let $f'(0) = 1$. Let M be the maximum value of the function $|f'(z)|(1-|z|)$, attained at point p. Note this maximum cannot occur on the boundary of the unit disk, as $|f'(z)|(1-|z|) = 0$ if $|z| = 1$, and at $z = 0$, we have that $|f'(0)|(1-|0|) = 1 > 0$.

Now we define a radius r so that the disk $D(p; r)$ is contained in $\mathbb{D}$. In particular, we can choose $r = \frac{1}{2}(1-|p|)$. From this we also have that $1 - |z| \geq r$ for every $z \in D(p; r)$.

Now from our definition of r, we have that $M = |f'(p)|(1-|p|) = 2r|f'(p)|$. Since M is the maximum value of this function, we must have that $|f'(z)|(1-|z|) \leq 2r|f'(p)|$ for all $z \in \overline{\mathbb{D}}$. Hence

$$|f'(z)| \leq \frac{r}{1-|z|}2|f'(p)| \leq 2|f'(p)|$$

for all $z \in D(p; r)$.

We can now apply Lemma 1.2.2 to f to get that $D(f(p); R) \subseteq f(D(p; r)) \subseteq f(\mathbb{D})$, where $R = (3-2\sqrt{2})r|f'(p)| = (\frac{3}{2} - \sqrt{2})M$. So the image of f contains a disk of radius R, where

$$R = (\tfrac{3}{2} - \sqrt{2})M \geq (\tfrac{3}{2} - \sqrt{2})|f'(0)|(1-|0|) = \tfrac{3}{2} - \sqrt{2}$$

□

We can generalise this statement a little bit. If we let f' have other non-zero values and allow f to be holomorphic on arbitrary open sets, we get a slightly more generalised version of Bloch's Theorem.

Corollary 1.2.3. *If f is holomorphic on some open set U, and that $f'(c) \neq 0$ for some $c \in U$. Let r be the distance from c to the boundary of U. Then $f(U)$ contains a disk of radius $\frac{1}{12}r|f'(c)|$.*

Proof. Firstly, we may assume without loss of generality that $c = 0$ (as we can define a function $g(z) = f(z + c)$ and continue this argument using g). Note that this means that f is holomorphic on $D(0; r) \subseteq U$. Now define h as

$$h(z) := \frac{f(rz)}{rf'(0)}$$

Clearly we have that the domain of h will include $\mathbb{D}$, and we moreover have that $h'(0) = 1$. So applying Bloch's Theorem to h, we get that $h(\mathbb{D})$ contains a disk of radius $\frac{3}{2} - \sqrt{2} > \frac{1}{12}$. This means that $f(U)$ will contain a disk of radius $\frac{1}{12} r|f'(0)|$. $\square$

One nice consequence of this statement is that we immediately have an alternative proofs of Liouville's Theorem. This is somewhat unsurprising however this theorem would also immediately follow from Picard's Little Theorem.

Theorem 1.2.4 (Liouville's Theorem). *Let f be a non-constant holomorphic function on $\mathbb{C}$. Then $|f|$ is unbounded.*

Proof. Since f is non-constant, there must be some point $c \in \mathbb{C}$ such that $f'(c) \neq 0$. Take any $r \in \mathbb{R}$. Trivially f is holomorphic on $D(c; r)$. So by Corollary 1.2.3, $f(U)$ (and hence $f(\mathbb{C})$ contains a disk of radius $\frac{1}{12}|f'(c)|r$. So in particular there must be some $z \in \mathbb{C}$ such that $|f(z)| > \frac{1}{13}|f'(c)|r$. Since we can choose r to be arbitrarily large, the theorem follows. $\square$

1.3 Picard's Little Theorem

We are close to the proof of Picard's Little Theorem. We need a few more lemmas that relate to cosine functions and then we will have completed the proof. In particular the way we shall prove this is to show that any (non-constant holomorphic) function which omits two values must also omit some kind of grid points. This would then give us a bound on the size of the disks in the image, which would contradict Bloch's Theorem.

We start by first giving sufficient conditions for a function to have a well defined 'logarithm' on some region U.

Theorem 1.3.1. *Let f be a function that is holomorphic on a simply connected region U. Suppose f has no zeroes in U. Then there is a holomorphic function g such that $f(z) = e^{g(z)}$ for all $z \in U$.*

Proof. First fix some point $z_0 \in U$. For each $z \in U$, let γ_z be a curve from z_0 to z, and define

$$F(z) := \int_{\gamma_z} \frac{f'(w)}{f(w)} \, \mathrm{d}w$$

Note that f has no zeroes so $f'(z)/f(z)$ is holomorphic in U. This means by Cauchy's theorem that $F(z)$ will be independent of the chosen path γ, and is therefore well defined. We can also see immediately from the definition that $F'(z) = f'(z)/f(z)$.

Now from the definition of F we have that

$$\begin{aligned} \left(\frac{f}{e^F}\right)' &= \frac{f'e^F - fF'e^F}{e^{2F}} \\ &= \frac{f' - fF'}{e^F} \\ &= 0 \end{aligned}$$

as $f'(z) = f(z)F'(z)$.

This means that the function f/e^F is constant. So in particular if we let $k \in \mathbb{C}$ such that $e^k = f(z_0)$, and if we define

$$g(z) := kF(z)$$

then

$$\frac{f(z)}{e^{g(z)}} = \frac{f(z)}{e^{kF(z)}} = \frac{f(z)}{f(z_0)e^{F(z)}} = \frac{f(z_0)}{f(z_0)e^{F(z_0)}} = \frac{1}{e^{F(z_0)}} = 1$$

and so

$$f(z) = e^{g(z)}$$

□

This then allows us to define a 'square root' on such an f, and with some stronger conditions, we can define an 'inverse cosine' as well.

Corollary 1.3.2. *Let f be a function that is holomorphic on a simply connected region U. Suppose f has no zeroes in U. Then there is a holomorphic function g such that $f(z) = g(z)^2$ for all $z \in U$.*

Proof. First let F be a holomorphic function such that $f(z) = e^{F(z)}$ for all $z \in U$. Let $g(z) = e^{F(z)/2}$. Then it follows that $g(z)^2 = e^{F(z)} = f(z)$. □

Lemma 1.3.3. *Let U be simply connected region and let f be holomorphic on U. Suppose that f is not 1 or -1 anywhere on U. Then there is a holomorphic function F on U such that*

$$f = \cos F$$

Proof. First we have that $1 - f^2$ contains no zeroes in U. So we can write $1 - f^2 = g^2$ for some holomorphic function g. This means that $(f + ig)(f - ig) = f^2 + g^2 = 1$, so in particular $f + ig$ does not have any zeroes. This means we can write $f + ig = e^{iF}$ for some holomorphic F. We can then also see that

$$e^{-iF} = \frac{1}{f + ig} = f - ig$$

and hence we have that

$$f = \frac{1}{2}\left[(f + ig) + (f - ig)\right] = \frac{1}{2}\left[e^{iF} + e^{-iF}\right] = \cos F$$

□

We will also need a more computational lemma:

Lemma 1.3.4. *If $z = x + iy$, then*

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

Proof. This can be evaluated directly:

$$\begin{aligned}
\cos x \cosh y - i \sin x \sinh y &= \frac{e^{ix} + e^{-ix}}{2}\frac{e^{y} + e^{-y}}{2} - i\frac{e^{ix} - e^{-ix}}{2i}\frac{e^{y} - e^{-y}}{2} \\
&= \frac{1}{4}\left[2e^{ix-y} + 2e^{-ix+y}\right] \\
&= \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} \\
&= \cos(x + iy) = \cos z
\end{aligned}$$

□

This leads us to the final significant lemma before the proof of Picard's Theorem. As mentioned prior, we take any function which omits two values, and by writing it as two nested cosine functions, we shall show that it must omit also a grid of points, and will hence have an upper bound on the disk size.

Lemma 1.3.5. *Let U be a simply connected region, and let f be holomorphic on U. Suppose that f is not 0 or 1 anywhere on U. Then there is a holomorphic function g on U such that*

$$f = \frac{1}{2}\left[1 + \cos\pi\left(\cos\pi g\right)\right]$$

Moreover, $g(U)$ contains no disk of radius 1.

Proof. We first show that such a function g exists. If we first consider the function $2f - 1$, we see that it does not attain values of 1 and -1. So by the previous lemma, we have that $2f - 1 = \cos\pi F$. Moreover, since $2f - 1$ does not attain 1 and -1, then we see that F cannot be an integer (otherwise $\cos\pi F = \pm 1$). So F also does not attain 1 and -1, so we can apply the previous lemma again to get that there is a holomorphic g such that $F = \cos\pi g$.

We next need to show that $g(U)$ contains no disk of radius 1. To do this, first define the set $A := \{m \pm i\pi^{-1}\ln(n + \sqrt{n^2 - 1}) : m, n \in \mathbb{N}\}$. This set A can be seen as a "grid" of points in $\mathbb{C}$. In particularly, as we shall show, no point in $\mathbb{C}$ will ever be further than a distance of 1 from some point in A.

Firstly, we can see that in the real direction, these, points are spaced equally apart with distance 1. So it follows immediately that for any $z \in C$, there must be some grid point w such that $|\mathrm{Re}(z) - \mathrm{Re}(w)| \le 1/2$. In fact, we can choose w so that the same is true in the imaginary direction. To see this, consider the vertical distance between two adjacent points:

$$\begin{aligned}
&\ln\left(n + 1 + \sqrt{(n+1)^2 - 1}\right) - \ln\left(n + \sqrt{n^2 - 1}\right)\\
&= \ln\left(\frac{n + 1 + \sqrt{n^2 + 2n)}}{n + \sqrt{n^2 - 1}}\right)\\
&= \ln\left(\frac{1 + \frac{1}{n} + \sqrt{1 + \frac{2}{n}}}{1 + \sqrt{1 - \frac{1}{n^2}}}\right)\\
&\le \ln\left(1 + \frac{1}{n} + \sqrt{1 + \frac{2}{n}}\right)\\
&\le \ln(2 + \sqrt{3}) < \ln\left(2^3\right) < \ln\left(e^\pi\right) = \pi
\end{aligned}$$

Note that we do not run into any issues when we change from positive imaginary components to negative imaginary components as when $n = 1$, we will have that the imaginary component is $\pm i\pi^{-1}\ln(n + \sqrt{n^2 - 1} = 0$.

This then tells us that any two consecutive grid points in the imaginary direction must be less than a distance of 1 apart. So for any $z \in \mathbb{C}$ we can find a $w \in A$ such that we have $|\mathrm{Re}(z) - \mathrm{Re}(w)| \le \frac{1}{2}$ and $|\mathrm{Im}(z) - \mathrm{Im}(w)| \le \frac{1}{2}$, and so $|z - w| < 1$.

To complete the proof then, we lastly need to show that g never maps onto any element in A. So take $w \in A$ and let $w = p \pm i\pi^{-1}\ln(q + \sqrt{q^2 - 1})$, and suppose that $g(z) = w$ for some $z \in U$. Then it would follow that

$$
\begin{aligned}
\cos \pi w &= \cos(\pi p) \cosh \ln\left(q + \sqrt{q^2-1}\right) - i \sin(\pi p) \sinh \ln\left(q + \sqrt{q^2-1}\right) \\
&= \cos(\pi p) \cosh \ln\left(q + \sqrt{q^2-1}\right) \\
&= \frac{1}{2}\cos(\pi p)\left[\left(q + \sqrt{q^2-1}\right)^{-1} + \left(q + \sqrt{q^2-1}\right)\right] \\
&= \frac{1}{2}\cos(\pi p)\left[\frac{q - \sqrt{q^2-1}}{q^2-(q^2-1)} + \left(q + \sqrt{q^2-1}\right)\right] \\
&= \frac{1}{2}\cos(\pi p)\left[q - \sqrt{q^2-1} + q + \sqrt{q^2-1}\right] \\
&= (-1)^p q
\end{aligned}
$$

This gives us that $\cos(\pi \cos \pi g(z)) = \pm 1$, so $f(z) = 0, 1$. This is contrary to our assumption, so $A \cap g(U) = \emptyset$. This means that $g(U)$ contains no disk of size 1, as any point in $g(U)$ is less than a distance of 1 away from a point not in $g(U)$. $\square$

If we now apply the previous lemma to a function on the entire complex plane, and then also use Bloch's Theorem, we finally obtain Picard's Little Theorem.

Theorem 1.3.6 (Picard's Little Theorem)**.** *Every non-constant holomorphic function on $\mathbb{C}$ omits at most one complex value in its image.*

Proof. Suppose that f is holomorphic on $\mathbb{C}$. Suppose also that there are distinct a, b such that $a, b \notin f(\mathbb{C})$. Define

$$h(z) := \frac{f(z) - a}{b - a}$$

We can see that then it follows that $0, 1 \notin h(\mathbb{C})$. We can then use Lemma 1.3.5 write h as

$$h(z) = \frac{1}{2}\left[1 + \cos \pi \left(\cos \pi g\right)\right]$$

where $g(\mathbb{C})$ has no disk of radius 1.

Now, suppose that g is non-constant, so that there is some $c \in \mathbb{C}$ such that $g'(c) \neq 0$. We can then use Bloch's Theorem to derive a contradiction. If we restrict the domain of g to $D(c; 12/|g'(c)|)$, then by applying Corollary 1.2.3, we see that g must contain a disk of radius 1. This is a contradiction, and so, g must be constant. Hence we have that h, and therefore f must also be constant. $\square$

Like a significant portion of other theorems we have encountered so far, this very much contrasts the real case. More than this however it is a very surprising statement even without the comparison. In fact, this is as strong as such a statement can be, as there do exist functions that omit a single value (such as the exponential which omits 0). We do obtain a slight variation as a corollary of the Great Picard Theorem (see Corollary 2.4.3)

One neat corollary of the Little Theorem, is the following fixed point theorem.

Proposition 1.3.7. *Let f be holomorphic on $\mathbb{C}$. Then $f \circ f$ always has a fixed point, unless f is a translation (of the form $f(z) = z + b$ for some $b \in \mathbb{C} \setminus \{0\}$).*

Proof. Suppose $f \circ f$ has no fixed points, which means f has no fixed points either. Define

$$g(z) := \frac{f(f(z)) - z}{f(z) - z}$$

Note g is holomorphic as f has no fixed points. Moreover, as both f and $f \circ f$ have no fixed points, it follows that g is never 0 or 1. So by Picard's theorem g must be a constant. Call this constant c, and note that $c \neq 0, 1$. This means we have that

$$f(f(z)) - z = c(f(z) - z)$$

Implicitly differentiating both sides we get that

$$\begin{aligned} f'(z)f'(f(z)) - 1 &= c(f'(z) - 1) \\ f'(z)[f'(f(z)) - c] &= 1 - c \end{aligned}$$

As $c \neq 1$, the right side of the above equation is never 0. So this means that $f'(z)$ is never 0, and that $f'(f(z))$ is never c. Since $c \neq 0$, this means that the function $f' \circ f$ omits 0 and c, so must be constant. Now, the image $f(\mathbb{C})$ is open (and in fact by Picard omits at most one point), so it follows that f' is constant. Therefore $f(z) = az + b$ for some $a, b \in \mathbb{C}$ and since f has no fixed points, then $a = 1$, $b \neq 0$. $\square$

Chapter 2

Picard's Great Theorem

We now turn our attention to the other of Picard's Theorems. This theorem is similar, though we require a few more results to prove it. The theorem itself states that any holomorphic function omits at most one value in any punctured neighbourhood of an essential singularity. This in fact implies that all values it attains, it must attain them infinitely often (we can just take successively smaller neighbourhoods).

While we will not build directly on the Little Theorem, we shall use and improve upon some of the results we saw from the previous section.

2.1 Arzelà-Ascoli & Montel

We first begin by examining sequences of holomorphic functions. In particular we shall look at what conditions of a sequence are needed so that it has a convergent subsequence. The reason why we shall need this will be more apparent in the proof of Picard's Great Theorem.

To begin, we have the following definition.

Definition 2.1.1. *Let X and Y be metric spaces. Let $f_n : X \to Y$ be a sequence of functions. The sequence f_n is said to be **equicontinuous** if for every $x \in X$ and $\varepsilon > 0$, there is some $\delta > 0$ such that if $d_X(x, y) < \delta$, then $d_Y(f_n(x), f_n(y)) < \varepsilon$ for every n.*

In essence, this definition says that each function in this sequence is continuous at the same rate.

Using this definition we get the theorem below. Note that the functions need only be continuous, not holomorphic, and moreover we work on a general compact metric space.

Theorem 2.1.2 (Arzelà-Ascoli)**.** *Let X be a compact metric space. Let $f_n : X \to \mathbb{C}$ be a pointwise bounded sequence of equicontinuous functions. Then f_n contains a subsequence that converges uniformly.*

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$ be a countable dense set in X. Such a set exists as X is compact.

Now define $g_{i,j}$ as follows. Let $g_{0,n} = f_n$, and let $(g_{i,n})_n$ be a subsequence of $(g_{i-1,n})_n$ so that $(g_{i,n}(a_n))_n$ converges. Such a subsequence converges as f_n is pointwise bounded. In other words, the subsequence $(g_{i,n})_n$ is chosen so that it converges at the points $a_1, a_2, \ldots, a_n$. If we now define $g_n = g_{n,n}$, then we obtain a subsequence of f_n that will eventually converge at all elements of A.

We now want to show that g_n converges uniformly on X. So let $\varepsilon > 0$, and choose any $x \in X$. Since f_n is equicontinuous, then there is a $\delta > 0$ such that $|g_n(x) - g_n(y)| < \varepsilon/3$ for every y such that $|x - y| < \delta$. Since A is dense, choose $a \in A$ so that $|a - x| < \delta$.

Now, g_n converges at a (as $a \in A$). In particular this means that $(g_n(a))_n$ is a Cauchy sequence. It then follows that there is N such that for all $n, m > N$

$$|g_n(a) - g_m(a)| \leq \frac{\varepsilon}{3}$$

It then follows that for any $n, m > N$, then

$$\begin{aligned}|g_n(x) - g_m(x)| &\leq |g_n(x) - g_n(a)| + |g_n(a) - g_m(a)| + |g_m(a) - g_m(x)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon\end{aligned}$$

This means that g_n is (uniformly) Cauchy at every point in X. So define

$$g(x) = \lim_{n\to\infty} g_n(x)$$

It then follows that for any $x \in X$ and $\varepsilon > 0$, if we choose N as above, then for any $n > N$ we will have that

$$\begin{aligned}|g(x) - g_n(x)| &= \left|\lim_{m\to\infty} g_m(x) - g_n(x)\right| \\ &= \lim_{m\to\infty} |g_m(x) - g_n(x)| \\ &\leq \varepsilon\end{aligned}$$

Where the last inequality holds as eventually $|g_m - g_n(x)| \leq \varepsilon$.

Hence g_n converges uniformly to g on X. □

In fact, this theorem can be further generalised by replacing $\mathbb{C}$ with a more general metric space with some conditions. However we will only be looking at holomorphic functions, so for simplicity we only consider $\mathbb{C}$.

We know aim to see what kind of sequence of holomorphic functions will be equicontinuous, so that they will satisfy the conditions of the above theorem. With that in mind, we have the following definition.

Definition 2.1.3. *Let X and Y be metric spaces. A sequence f_n of functions from X to Y is said to be* ***locally bounded*** *(in X) if for every $x \in X$, there is some open neighbourhood $U_x \subseteq X$ of x such that every f_n is bounded in U by the same bound. In other words, there is some U_x and M such that for every n*

$$diam_Y(f_n(U_x)) \leq M_x$$

where $diam_Y(A)$ is the diameter of $A \subseteq Y$.

In the case where $Y = \mathbb{C}$, then this equivalent to saying that there is an M'_x so that $|f_n(z)| \leq M'_x$ for every $n \in \mathbb{N}$ and $x \in U_x$.

We can see immediately from the definition that a locally bounded sequence is pointwise bounded. As it turns out, if our sequence is holomorphic, then it will also be equicontinuous.

Lemma 2.1.4. *Let f_n be a locally bounded sequence of functions holomorphic in U. Then for every $c \in U$, and $\varepsilon > 0$, there is a disk $D \subseteq U$ about c such that*

$$|f_n(w) - f_n(z)| \leq \varepsilon$$

for every $n \in \mathbb{N}$ and $w, z \in D$.

Proof. Firstly, choose $r > 0$ small enough so that $\hat{D} = D(c; 2r) \subseteq U$, and moreover so that f_n is bounded by M on $\hat{D}$. Also define $D' = D(c; r)$. From the Cauchy integral formula, it follows that

$$\begin{aligned} f_n(w) - f_n(z) &= \frac{1}{2\pi i} \int_{\partial \hat{D}} \left(\frac{f_n(\zeta)}{\zeta - w} - \frac{f_n(\zeta)}{\zeta - z} \right) \, \mathrm{d}\zeta \\ &= \frac{w - z}{2\pi i} \int_{\partial \hat{D}} \frac{f_n(\zeta)}{(\zeta - w)(\zeta - z)} \, \mathrm{d}\zeta \end{aligned}$$

for every $w, z \in D'$.

Since ζ is on the boundary of $\hat{D}$, then ζ and w will be at least a distance of r apart (as well as z). So we have $|\zeta - w||\zeta - z| \geq r^2$. This allows to obtain the following bound:

$$|f_n(w) - f_n(z)| \leq \frac{|w - z|}{2\pi} \frac{|f_n|_{\hat{D}}}{r^2} 2\pi r = |w - z||f_n|_{\hat{D}} \frac{1}{r} \leq |w - z| \frac{M}{r}$$

Now for any given $\varepsilon > 0$, let $\delta = \min\{r, \varepsilon r / 2M\}$. Now define the disk $D = D(c; \delta)$. Since $\delta \leq r$, the inequality above holds for every $z, w \in D$. Moreover, we will have that $|w - z| \leq 2\delta$ in D, so that

$$|f(w) - f(z)| \leq \frac{2\delta M}{r} \leq \varepsilon$$

□

It now follows by Arzelà-Ascoli that any sequence of locally bounded holomorphic functions on a compact set will have a convergent subsequence. However we do not know if this function will also be holomorphic, let alone continuous. We also want to look at sequences defined on non-compact domains.

The following definition gives a notion of convergence which is similar to uniform convergence on non-compact domains.

Definition 2.1.5. *Let $f_n : X \to Y$ be a sequence of functions and $f : X \to Y$. The sequence f_n is said to* ***converge compactly*** *to f if for every compact set $F \subseteq X$, f_n converges to f uniformly on F.*

With this, we shall see that if a (locally bounded, holomorphic) sequence converges compactly to a function f, then this f will indeed be continuous and moreover holomorphic. A stronger statement is contained in the following theorem.

Theorem 2.1.6. *Let f_n be a sequence of holomorphic functions on a region U. Suppose f_n converges compactly to a function f. Then f is holomorphic. Moreover we have that f_n' converges compactly to f'.*

Proof. First take any $x \in U$, and let E be a compact set in U containing x. So f_n converges uniformly to f in E. Since each f_n is continuous, then f must be continuous as well. In particular f is continuous at x. This is true for every $x \in U$, so f is continuous in U.

Take any triangle γ in U. Since the γ is compact, it follows that f_n converges to f compactly on γ. This means that f_n is eventually bounded on γ. It then follows that

$$\int_\gamma f \, \mathrm{d}z = \int_\gamma \lim_{n\to\infty} f_n \, \mathrm{d}z = \lim_{n\to\infty} \int_\gamma f_n \, \mathrm{d}z = 0$$

This means that the integral of f on every triangle in U is zero. Since f is continuous, by Morera's Theorem, f must be holomorphic in U.

To show that the derivatives also converge, we use Cauchy's integral formula for higher derivatives. We will show this for the first derivative only, but this can be generalised for higher derivatives (or one may use a simple induction argument).

Take any $w \in U$, and let r_w be small enough so that $\overline{D}(w; r_w) \subseteq U$. It is clear that as U is open, we can write

$$U = \bigcup_{w\in U} D\left(w; \frac{r_w}{2}\right)$$

Now take any compact set $E \subseteq U$. Obviously the above union of disks will cover E, so there is a finite subcover of E. That is,

$$E \subseteq \bigcup_{i=1}^{k} D\left(w_i; \frac{r_{w_i}}{2}\right)$$

for some $w_1, \ldots, w_k \in U$.

Let $\varepsilon > 0$. Take any w_i and let $r_i = r_{w_i}$, $D_i = D(w_i; r_i/2)$, and $C_i = C(w_i; r_i)$.

Now, f_n converges uniformly to f in C_i as C_i is compact. It then follows that there is some N_i so that for every $n > N_i$, we have $|f(z) - f_n(z)| < \varepsilon r_i/4$ for every $z \in C_i$.

By Cauchy's integral formula, we have that

$$f'(z) = \frac{1}{2\pi i} \int_{C_i} \frac{f(\zeta)}{(\zeta - z)^2} \, \mathrm{d}\zeta$$

for any $z \in D_i$. Now, note that $|\zeta - z| \geq r_i/2$ for any $\zeta \in C_i$. This means we have that

$$\begin{aligned}
|f'(z) - f_n'(z)| &= \frac{1}{2\pi} \left| \int_{C_i} \frac{f(\zeta) - f_n'(\zeta)}{(\zeta - z)^2} \, \mathrm{d}\zeta \right| \\
&\leq \frac{1}{2\pi} \frac{\varepsilon r_i/4}{(r_i/2)^2} 2\pi r_i \\
&= \varepsilon
\end{aligned}$$

In particular if we take $N = \max\{N_i : i = 1, \ldots, k\}$, then for any $z \in E$, we will have $z \in D_i$ for some i so that

$$|f'(z) - f_n'(z)| \leq \varepsilon$$

for every $n > N \geq N_i$.

Hence f_n' uniformly converges to f' on E, and thus compactly converges on U. □

We can finally tie everything together. Take f_n to be a locally bounded sequence of holomorphic functions. We have then seen that if we restrict the domain of f_n to a compact set, that it will have a convergent subsequence. We have also seen that if such a sequence has compactly converging subsequence, then the resulting function will also be holomorphic. All that remains is to show that the existence of a convergent subsequence on compact subsets will give us a compactly convergent subsequence.

This is contained in the following theorem of Montel.

Theorem 2.1.7 (Montel's Theorem)**.** *Let f_n be a locally bounded sequence of holomorphic functions on a region U. Then f_n has a subsequence that converges compactly to a holomorphic function.*

Proof. Since f_n is a locally bounded sequence, by Lemma 2.1.4, the sequence f_n is also equicontinuous. All that remains is to show that there is a subsequence that converges uniformly on every compact subset (of U) to a holomorphic function f.

Define the following sequence of sets

$$K_m = \{z \in \mathbb{C} : d(z, \mathbb{C} \setminus U) \geq \tfrac{1}{m} \text{ and } |z| \leq m\}$$

We can clearly see that each K_m is closed, and is bounded by the second condition. Therefore K_m is compact. Moreover we can also see that z is in some K_m precisely when it is a non-zero distance from $\mathbb{C} \setminus U$, and since U is open, we then have that

$$U = \bigcup_{m=1}^{\infty} K_m$$

Note that any compact subset of U will be a non-zero distance away from the boundary of U, so by definition will be contained in some K_m.

We now repeat a similar diagonalisation argument as in Arzelà-Ascoli to construct a convergent subsequence. In particular we first let $g_{0,n} = f_n$, then we let $(g_{i,n})_n$ be a subsequence of $(g_{i-1,n})_n$ which converges uniformly on K_i. Such a subsequence must exist by Arzelà-Ascoli.

If we then define $g_n = g_{n,n}$, then we see that eventually, g_n will converge uniformly on each K_m. So define

$$f := \lim_{n\to\infty} g_n$$

We then have that g_n is compactly convergent to f, as g_n converges uniformly to f in each K_m, and every compact subset of U is contained in some K_m.

Lastly, it follows from Theorem 2.1.6 that f is holomorphic. □

This is an incredibly powerful theorem, and we shall later look at what conditions we can make on a subsequence f_n to ensure that it is locally bounded.

We shall also note, that we have only been considering sequences of function. One may generalise this and look at entire families of functions, and obtain similar results. We shall only need this result for sequence however, so we shall continue to restrict our attention to this case.

2.2 Hurwitz's Theorem

We now take a brief aside to look at a corollary of Montel's Theorem. In particular it tells us about the nature of a sequence of non-vanishing functions. First we have the following lemma of Hurwitz.

Lemma 2.2.1 (Hurwitz's Lemma)**.** *Let f_n be a sequence of holomorphic functions that converge compactly to f on a region G. Suppose that f is not constant. Then there for any $c \in G$, there is some $N \in \mathbb{N}$ (depending on c) and $c_n \in G$ so that*

$$f_n(c_n) = f(c)$$

for every $n \geq N$.

Proof. Take any $c \in G$ and let $g_n(z) = f_n(z+c) - f(c)$, so that $g(z) = f(z+c) - f(c)$. We know that f is non-constant, so g is as well. We shall denote $G' = \{w - c : w \in G\}$ so that g and g_n are holomorphic on G'. Since f is non-constant, and $g(0) = 0$, this means that g is not identically zero on G'.

Now let $k \in \mathbb{N}$ and define $C = C(0, r) \subseteq G'$ so that g is non-vanishing on C. Such a C must exist, as otherwise we would have a convergent sequence of zeroes of g in G'. This would then mean that g would be identically zero on G', which is not the case.

Now, C is compact, and g is non-vanishing on C, so there is some $\delta > 0$ so that

$$\delta = \inf\{|g(z)| : z \in C\}$$

Moreover, g_n will converge uniformly to g, so for some N we will have that

$$|g(z) - g_n(z)| \leq \frac{1}{2}\delta$$

for all $n \geq N$, $z \in C$. It then follows that

$$|g(z) - g_n(z)| \leq \frac{1}{2}\delta < \delta \leq |g(z)| \leq |g(z)| + |g_n(z)|$$

It then follows by Rouché (Theorem 1.1.4) that for all $n \geq N$, that g and g_n have the same number of zeroes inside C. Since $g(0) = 0$, then these g_n must also have at least one zero inside C. So there is some d_n so that $g_n(d_n) = 0$, and hence $f_n(d_n + c) = f(c)$. □

In fact, it would follow that $d_n \to 0$, (so that $c_n \to c$), and one may see this is the case by taking smaller and smaller circles inside G'. We shall however omit the details as we do not need this fact.

From this lemma we obtain the corollary about non-vanishing functions. We shall need this later.

Corollary 2.2.2. *Let f_n be a sequence of non-vanishing holomorphic functions that converge compactly to a function f. If f is not identically zero, then f is non-vanishing.*

Proof. Suppose f is non-constant. If f had a zero, then by Lemma 2.2.1, there is a sequence c_n and N so that $f_n(c_n) = 0$. This means that the f_n are vanishing at some point and so we have a contradiction. □

Another result we obtain is the following theorem about injective functions.

Theorem 2.2.3 (Hurwitz's Injection Theorem)**.** *Let G and G' be regions and let $f_n : G \to G'$ be a sequence of functions that converge compactly to a non-constant holomorphic function on G. If each f_n is injective, then so is f and moreover $f(G) \subseteq G'$.*

Proof. We first prove the injectivity of f. Take $c \in G$, and let $g_n(z) := f_n(z) - f_n(c)$ so that g_n converges compactly to $g(z) = f(z) - f(c)$ on $G \setminus \{c\}$. Since f_n is injective, then g_n is non-vanishing on $G \setminus \{c\}$. Moreover since f is non-constant, then g is not identically zero. By Corollary 2.2.2 it follows that g is non-vanishing $G \setminus \{c\}$, so that if $z \neq c$, then $f(z) \neq f(c)$. Hence f is injective on G.

The proof that $f(G) \subseteq G'$ is very similar. Take any $d \notin G'$, and let $h_n(z) := f_n(z) - d$ so that h_n converges compactly to $h(z) = f(z) - d$ on G. Since $d \notin G'$, then each h_n is non-vanishing. Moreover since f is non-constant, then h is not identically zero. By Corollary 2.2.2 this means that h is non-vanishing and so $d \notin f(G)$. Hence $f(G) \subseteq G'$. □

We will use this theorem when we look at injective functions and the Riemann Mapping Theorem in Chapter 3.

2.3 Schottky's Theorem

We shall look at improving our condition of local boundedness. In particular, as we shall see, omitting two values from the image will be sufficient to guarantee something similar. To do this, we shall look at providing an upper bound for function defined on a disk. The existence of such a bound is known as Schottky's Theorem.

Naturally, we shall need some inequalities.

Lemma 2.3.1. *For any $w \in \mathbb{C}$, we have that $|\cos w| \leq e^{|w|}$ and that $\frac{1}{2}|1 + \cos w| \leq e^{|w|}$.*

Proof. We begin with the first inequality. We shall show the square of the left side is smaller than the square of the right. So let $w = x + yi$, so that

$$\left(e^{|w|}\right)^2 = e^{2|w|} = e^{2\sqrt{x^2+y^2}} \geq e^{2|y|}$$

We can also see that the left-hand side is

$$\begin{aligned}
|\cos(w)|^2 &= \frac{1}{4}\left(e^{iw} + e^{-iw}\right)\left(e^{i\bar{w}} + e^{-i\bar{w}}\right) \\
&= \frac{1}{4}\left(e^{i(w+\bar{w})} + e^{i(w-\bar{w})} + e^{-i(w-\bar{w})} + e^{-i(w+\bar{w})}\right) \\
&= \frac{1}{4}\left(e^{2xi} + e^{2y} + e^{-2y} + e^{-2xi}\right) \\
&= \frac{1}{4}\left(e^{2y} + e^{-2y} + 2\cos(2x)\right)
\end{aligned}$$

Now we have that $2|y| \geq 2y, -2y, 0$. The last inequality means that $e^{2|y|} \geq 1$. Therefore we have that

$$4e^{2|y|} \geq e^{2y} + e^{-2y} + 2 \geq e^{2y} + e^{-2y} + 2\cos(2x)$$

and so it follows that

$$\left(e^{|w|}\right)^2 \geq e^{2|y|} \geq \frac{1}{4}\left(e^{2y} + e^{-2y} + 2\cos(2x)\right) = |\cos(w)|^2$$

The second inequality follows from the first

$$\begin{aligned}
|1+\cos w|^2 &\leq 1+2|\cos w|+|\cos w|^2 \\
&\leq e^{2|w|}+2e^{|w|}+\left(e^{|w|}\right)^2 \\
&\leq 4e^{2|w|}
\end{aligned}$$

Hence

$$\frac{1}{2}|1+\cos w| \leq e^{|w|}$$

□

We also have the following statement about cosines.

Lemma 2.3.2. *Let $w \in \mathbb{C}$. Then there is a $v \in \mathbb{C}$ such that $\cos \pi v = w$ and $|v| \leq 1 + |w|$.*

Proof. For any w, we can always find a v such that $\cos \pi v = w$. In particular, choose a v such that if we write $v = x + yi$, then $|x| \leq 1$. We can always choose such a x, because if $v' = x' + yi$ is a solution, then so is $v' + 2n$ for every $n \in \mathbb{Z}$.

Now for convenience, let $\alpha = \pi x$ and $\beta = \pi y$. Then we have that

$$\begin{aligned}
|w|^2 &= \left|\frac{e^{i\alpha-\beta}+e^{-i\alpha+\beta}}{2}\right|^2 \\
&= \frac{1}{4}\left(e^{i\alpha-\beta}+e^{-i\alpha+\beta}\right)\left(e^{-i\alpha-\beta}+e^{i\alpha+\beta}\right) \\
&= \frac{1}{4}\left[e^{-2\beta}+e^{2i\alpha}+e^{-2i\alpha}+e^{2\beta}\right] \\
&= \frac{1}{4}\left[e^{2\beta}-2+e^{-2\beta}\right]+\frac{1}{4}\left[e^{2i\alpha}+2+e^{-2i\alpha}\right] \\
&= \sinh^2\beta+\cos^2\alpha \\
&\geq \sinh^2\beta \\
&\geq \beta^2
\end{aligned}$$

This means we have that $y^2 \leq |w|^2/\pi^2$, as well as $x^2 \leq 1$. This means that

$$|v| = \sqrt{x^2+y^2} \leq \sqrt{1+|w|^2/\pi^2} \leq 1+|w|$$

□

We now get to the lemma where the heart of the proof in Schottky's Theorem lies. In fact, (somewhat unsurprisingly) we see that this lemma is very similar to Lemma 1.3.5 for which we used to prove Picard's Little Theorem. However, our approach this time shall be different, as our domain is now the unit disk, and we shall be trying to obtain an upper bound on our function g.

This is detailed in the following lemma.

Lemma 2.3.3. *Let f be holomorphic on the closed unit disk $\overline{\mathbb{D}}$, and suppose $0, 1 \notin f(\mathbb{D})$. Then there is a function g such that:*

1. $f = \frac{1}{2}\left[1+\cos\pi\left(\cos\pi g\right)\right]$
2. $|g(0)| \leq 3 + 2|f(0)|$
3. $|g(z)| \leq |g(0)| + 12\eta/(1-\eta)$ *for every* z, η *such that* $|z| \leq \eta$ *and* $0 < \eta < 1$.

Proof. The first point follows immediately from Lemma 1.3.5. However we will construct g explicitly again to obtain the bound in the second point.

Since $1-2f$ does not attain ± 1, then we have that $\cos\pi\bar{F} = 2f-1$ for some holomorphic function $\bar{F}$. By Lemma 2.3.2, it follows that there is some $b \in \mathbb{C}$ such that $\cos\pi b = 2|f(0)| - 1$, and $|b| \leq 1 + |2f(0)-1| \leq 2|f(0)| + 2$. Moreover, since $\cos\pi b = \cos\pi\bar{F}(0)$, then we have that $b = \pm\bar{F}(0) + 2k$ for some $k \in \mathbb{Z}$. Now let $F = \pm\bar{F} + 2k$, so that $F(0) = b$, and $2f - 1 = \cos\pi F$.

Now, F omits all integers, so we have that $F = \cos\bar{g}$ for some $\bar{g}$. Repeating this process, we see that there is some $a \in \mathbb{C}$ such that $\cos\pi a = b$ and $|a| \leq 1 + |b| \leq 3 + 2|f(0)|$. And since $\cos\pi a = \cos\pi\bar{g}(0)$, then again we have that $a = \pm\bar{g}(0) + 2n$ for some $n \in \mathbb{Z}$. So let $g = \pm\bar{g} + 2n$, so that $g(0) = a$ and $F = \cos\pi g$.

This means we have that

$$\frac{1}{2}\left[1+\cos\pi\left(\cos\pi g\right)\right] = \frac{1}{2}\left[1+\cos\pi F\right] = f$$

and

$$|g(0)| = |a| \leq 1 + |b| \leq 3 + 2|f(0)|$$

For the third point, let $0 < \eta < 1$, and take any z such that $|z| \leq \eta$. Note that the distance from z to the boundary of $\mathbb{D}$ is at least $1-\eta$. By Corollary 1.2.3, this means that g contains a disk of radius $\frac{1}{12}(1-\eta)|g'(z)|$. But, by Lemma 1.3.5 we know that g has no disk of radius 1. So it follows that $\frac{1}{12}(1-\eta)|g'(z)| \leq 1$.

This means we have that

$$\begin{aligned}
|g(z)| - |g(0)| &\leq |g(z) - g(0)| \\
&= \left| \int_{[0,z]} g'(u)\ \mathrm{d}u \right| \\
&\leq \left| \int_{[0,z]} \frac{12}{1-\eta}\ \mathrm{d}u \right| \\
&\leq \frac{12\eta}{1-\eta}
\end{aligned}$$

And so we finally have that

$$|g(z)| \leq |g(0)| + \frac{12\eta}{1-\eta}$$

□

Note that the 12 that we obtain in $12\eta/(1-\eta)$ comes directly from Bloch's Theorem. We could in fact use $1/B$ for any B which is less that Bloch's constant. However, choosing $B = 1/12$ shall suffice.

Next define the function L to be

$$L(\eta, r) := \exp\left[\pi\exp\left[\pi\left(3 + 2r + \frac{12\eta}{1-\eta}\right)\right]\right] \tag{2.1}$$

From this definition, we see that L bounds certain functions on the unit disk.

Theorem 2.3.4 (Schottky's Theorem). *Let f be holomorphic on $\overline{\mathbb{D}}$. Suppose that $0, 1 \notin f(\overline{\mathbb{D}})$, and that $|f(0)| \leq r$. Then $|f(z)| \leq L(\eta, r)$ for all z such that $|z| \leq \eta$, and $0 < \eta < 1$.*

Proof. Let g be the function described in the previous lemma. Take any $0 < \eta < 1$ and any z such that $|z| \leq 1$. Using the properties of g, as well as the inequalities in Lemma 2.3.1, we have that

$$\begin{aligned}
|f(z)| &= \frac{1}{2}\left|1 + \cos\pi\left(\cos\pi g(z)\right)\right| \\
&\leq \exp\left[\pi\left|\cos\pi g(z)\right|\right] \\
&\leq \exp\left[\pi\exp\pi\left|g(z)\right|\right] \\
&\leq \exp\left[\pi\exp\pi\left(|g(0)| + \frac{12\eta}{1-\eta}\right)\right] \\
&\leq \exp\left[\pi\exp\pi\left(3 + 2|f(0)| + \frac{12\eta}{1-\eta}\right)\right] \\
&\leq \exp\left[\pi\exp\pi\left(3 + 2r + \frac{12\eta}{1-\eta}\right)\right] = L(\eta, r)
\end{aligned}$$

□

From this we get the following corollary.

Corollary 2.3.5. *Let f_n be a sequence of holomorphic functions on a region U, such that $0, 1 \notin f(U)$ for every f_n. Let $w \in U$, and suppose there is some $r \in (0, \infty)$ such that $|f_n(w)| \leq r$. Then the sequence f_n is bounded on some neighbourhood of w.*

Proof. Take a closed disk $\overline{D} = \overline{D}(w; 2t) \subseteq U$. We can assume without loss of generality that $2t = 1$ and $w = 0$ (to do this one could consider g holomorphic on the unit disk defined by $g(z) = f((z-w)/2t)$). Then by Schottky it follows that for all $n \in \mathbb{N}$ and $z \in D(w; t)$, that

$$|f_n(z)| \leq L\left(\tfrac{1}{2}, r\right)$$

□

To get local boundedness on the entire region U, we shall need something stronger. In fact, we show that the at no point in U that it can 'suddenly' go from being locally bounded to not. Then using connectedness we see that our sequence must be locally bounded everywhere in U.

This is made precise in the following theorem.

Theorem 2.3.6. *Let f_n be a sequence of holomorphic functions on a region U, such that $0, 1 \notin f(U)$. Let $w_0 \in U$, and suppose there is some $r \in (0, \infty)$ such that $|f_n(w_0)| \leq r$ for every f_n. Then the sequence f_n is locally bounded in U.*

Proof. Let

$$V = \{z \in U : f_n \text{ is bounded in some neighbourhood of } w\}$$

We can clearly see that V is open - we can simply write V as the union of open sets on which f_n is bounded. We can also see that $w_0 \in V$ by the previous lemma.

Now suppose that $V \neq U$. Since V is open in $\mathbb{C}$, and U is open the V is open in U. This means that V cannot be closed in U, as this would form a separation of U. So there must be some $w \in \partial_U V \setminus V$. Since $w \notin V$, then there is a subsequence f_{n_k} such that $\lim f_{n_k}(w) = \infty$. Now let $g_k = 1/f_{n_k}$ so that $\lim g_k(w) = 0$. Since $0, 1 \notin g_k(U)$, then by the previous lemma we have that g_k is bounded in a neighbourhood of w. So by Montel's Theorem, g_k has a subsequence g_{k_l} which converges uniformly to a holomorphic function g.

Now all g_k are non-vanishing, but $g(w) = 0$, so by Corollary 2.2.2, we have that g is identically zero. Since w is on the boundary of V, and g is defined on an open neighbourhood of V, then $g(u) = 0$ for some $u \in U$. But this is a contradiction as f_n is bounded in any neighbourhood of u.

So $V = U$, and the result follows. □

Lastly, we can in fact drop the condition that $|f_n(w)| \leq r$ at some point w. However in doing so we obtain a slightly weaker statement as follows.

Lemma 2.3.7. *Let f_n be a sequence of holomorphic functions on a region U, such that $0, 1 \notin f(U)$. Then either f_n or $1/f_n$ has a convergent subsequence.*

Proof. Let $g_n := 1/f_n$. Note that the g_n are holomorphic on U (as $0 \notin f_n(U)$) and moreover we have $0, 1 \notin g_n(U)$.

Now choose some $w \in U$ and let $N = \{n \in \mathbb{N} : |f_n(w)| \leq 1\}$. If N is infinite, then trivially there are infinitely many n such that $|f_n(w)| \leq 1$. Otherwise if N is finite, then there are infinitely many n such that $|g_n(w)| \leq 1$. Without loss of generality, suppose the former is true, and let f_{n_k} be a subsequence such that $|f_{n_k}(w)| \leq 1$. By Theorem 2.3.6, this means that f_{n_k} is locally bounded in U, and so by Montel's theorem, f_{n_k} has a subsequence that converges to a holomorphic function. □

2.4 Picard's Great Theorem

We are nearing the proof of Picard's Great Theorem. The majority of the work has been done in building up the statements in the previous sections, in particular the statement in Lemma 2.3.7. The following lemma uses this to give the proof of the theorem.

Lemma 2.4.1. *Suppose that f is holomorphic on the punctured unit disk, and suppose that 0 and 1 are not in the image of f. Then either f or $1/f$ are bounded in a (punctured) neighbourhood of 0.*

Proof. First define a sequence of functions $f_n(z) := f(z/n)$. By Lemma 2.3.7, there is some subsequence f_{n_k} such that either f_{n_k} or $1/f_{n_k}$ converges to a holomorphic function.

Suppose without loss of generality that the former case is true. In this case, the limit of f_{n_k} will be bounded on the circle $C(0, \frac{1}{2})$ (as it is compact). So we have that $|f_{n_k}| = |f(z/n_k)| \leq M$ for all n_k and z such that $|z| = \frac{1}{2}$. In other words we have that $|f(z)| \leq M$ for all z such that $|z| = 1/(2n_k)$.

This means that we have that f is bounded (by M) on concentric circles whose radius approaches zero. If we take any two of these circles, we an form an annulus. Now by the Maximum Principle, a holomorphic function can obtain a maximum on the interior of the domain. So by Corollary 1.1.8, we see that $|f(z)| \leq M$ for all z such that $1/(2n_{k+1}) \leq z \leq 1/(2n_k)$. Hence it follows that f is bounded on a punctured neighbourhood of 0. □

With this, we merely tidy up some loose ends and generalise the statement to get Picard's Great Theorem in full.

Theorem 2.4.2 (Picard's Great Theorem)**.** *Let $c \in \mathbb{C}$, and let f be a function holomorphic on an punctured open neighbourhood of c, and suppose moreover that f has an essential singularity at c. Then f assumes every complex number as a value infinitely many times, with at most one exception.*

Proof. Suppose that $a, b \in \mathbb{C}$ occur as a value of f in a neighbourhood of c only finitely many times. Since they only occur finitely many times, there must be some (punctured) open disk $D(c;r)$ such that $a, b \notin g(D(c;r))$. Now if we define

$$g(z) := \frac{f((z-c)/r) - a}{b-a}$$

we see that g does not attain the values 0 and 1 in the punctured unit disk. By the previous lemma this means that either g or $1/g$ is bounded in a punctured neighbourhood of 0.

If g is bounded, then it follows that the singularity at c is removable. Otherwise if $1/g$ is bounded, then $1/g$ has a removable singularity, and so c must either be a removable singularity or a pole of g. This means that the singularity at f is either removable or a pole, which is a contradiction.

Hence, we cannot have two complex numbers which only occur finitely many times as a value of f in a neighbourhood of c. $\square$

From this, we can also obtain the following stronger version of the Little Theorem.

Corollary 2.4.3. *Let f be a holomorphic function on $\mathbb{C}$. If f is not polynomial, then f assumes every complex number infinitely many times, with at most one exception.*

Proof. Define $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $g(z) = f(1/z)$. Clearly g is holomorphic on a punctured neighbourhood of 0. Moreover, since f is not a polynomial, 0 will be an essential singularity of g. So by Picard's Great Theorem, g will assume every complex number infinitely many times with at most one exception, and by definition so will f. $\square$

This is in fact a strengthening, as the case for polynomials can be seen as an application of Fundamental Theorem of Algebra. In fact, this would also tell us that any polynomial cannot possibly attain any given value infinitely many times (otherwise it would have infinitely many roots). This is a big difference between polynomials and non-polynomial holomorphic functions on $\mathbb{C}$.

Chapter 3

Riemann Mapping Theorem

We lastly turn to explore a different section of complex analysis. We will work towards proving the Riemann Mapping Theorem, which in essence tells us that all simply connected regions of $\mathbb{C}$, except $\mathbb{C}$ itself, are 'equivalent' in some sense to each other. More specifically, we mean that two regions are equivalent if there is a *biholomorphic* mapping between them.

3.1 Biholomorphic Functions

We begin with the definition of a biholomorphic function.

Definition 3.1.1. *A function $f : G \to H$ is **biholomorphic** if f is bijective and both f and f^{-1} are holomorphic (on G and H respectively).*

This definition captures the idea that we can transform G to H and back again in a sufficiently smooth way. Note that since we require that our mapping to be holomorphic, this is a lot stronger than simply being (real) differentiable. Finding such a mapping in general can be quite difficult.

We start with the following lemma about bijective holomorphic functions.

Lemma 3.1.2. *Let $f : G \to H$ be a bijective holomorphic function. Then f' is non-vanishing.*

Proof. Suppose that f' vanishes. Without loss of generality suppose that $f'(0) = f(0) = 0$. Since both of these are zero, we can easily see that f has at least a double zero at 0.

Now take r so that f is holomorphic on $D(0;r)$. Take any $c \in \mathbb{C}$ so that $0 < c < |f(z)|$ for all $z \in C = C(0;r/2)$. Note that such a c must exist as C is compact so $|f(z)|$ achieves a minimum, but f is never zero on C as f is injective.

Now define $f_c(z) := f(z) - c$. We can see that $|f(z) - f_c(z)| = |c| < |f(z)|$ for all $z \in C$. so it follows by Rouché that f and f_c share the same number of zeroes inside $D(0;r/2)$. Since f has a double zero, then f_c must have at least two zeroes, or a double zero. The former case cannot hold, as otherwise we would have that $f(z_1) - c = 0 = f(z_2) - c$, so $f(z_1) = f(z_2)$, contradicting the injectivity of f.

So f_c has a double zero. Let this zero be at z_c. We can see from our definition that $f_c(z_c) = 0$, and that since this a double zero, we will also have that $f_c'(z_c) = 0$. From this we can write

$$f'(z_c) = f_c'(z_c) = 0$$

And hence f' has a zero at z_c.

Now, each z_c must be distinct for different values of c. This means that f' is zero at infinitely many points in the closed disk $\overline{D}(0; r/2)$. Since this disk is compact it follows that f' is identically zero, and hence that f is constant.

This is a contradiction however, as f is bijective. $\square$

This lemma allows us to weaken the definition of biholomorphic functions. This is because requiring f^{-1} to be holomorphic turns out to be redundant.

Lemma 3.1.3. *Let $f : G \to H$ be a bijective holomorphic function. Then f is biholomorphic.*

Proof. Since f is holomorphic, it is an open map. So its inverse is continuous.

Now, let $g = f^{-1}$. By definition we have that

$$g'(z) = \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}$$

provided the limit exists. Let $w = g(z)$, and $w_0 = g(z_0)$. We can also see that $z = f(w)$ and $z_0 = f(w_0)$. Now since g is bicontinuous, we then have that

$$g'(z) = \lim_{w \to w_0} \frac{w - w_0}{f(w) - f(w_0)} = \frac{1}{f'(w_0)}$$

whenever $f'(w_0) \neq 0$. However we know that f' is non-vanishing, and therefore our inverse function g is holomorphic. $\square$

This theorem will help us in finding biholomorphic functions, as we only need to find bijective holomorphic functions. We have seen already several theorem for finding holomorphic functions (such as Montel's Theorem, which we shall later use), so we will focus more on finding bijective functions.

We also need to consider what makes simply connected domains useful in finding biholomorphic functions. When we prove that the Riemann Mapping Theorem, we shall not need to use any topological properties of being simply connected. We shall instead only consider domains defined as follows.

Definition 3.1.4. *Let $G \subseteq \mathbb{C}$ with $0 \in G$. Suppose that for every non-vanishing holomorphic function f on G there is a holomorphic function g such that $f = g^2$. Then we say that G is a **Q-domain**. Such a function g is called the **square-root of** f.*

Requiring $0 \in G$ is merely a convenience - any such region with $0 \notin G$ can simply be translated. If we do not require that $0 \in G$, we shall simply say that G has the **square-root property**. As it turns out, we have already seen that all simply connected regions have this property.

Lemma 3.1.5. *Let U be simply connected open region. Then U has the square-root property.*

Proof. Take any non-vanishing holomorphic function f. By Corollary 1.3.2, it follows immediately that there is a g such that $f = g^2$. $\square$

Somewhat unsurprisingly, the square-root property is invariant under biholomorphic images.

Corollary 3.1.6. *If $f : G \to H$ is biholomorphic and G has the square-root property. Then H has the square-root property.*

Proof. Let v be a non-vanishing holomorphic function on H. Define $u = v \circ f$, so that u is non-vanishing in G. Then there is a holomorphic function g on G such that $g^2 = u = v \circ f$. If we define $h = g \circ f^{-1}$, then

$$h^2 = g^2 \circ f^{-1} = u \circ f^{-1} = v \circ f \circ f^{-1} = v$$

□

3.2 Schwarz Lemma

This next lemma gives a useful statement about holomorphic functions mapping the unit disk onto itself. In essence it tells us that any such function which fixes 0 will either be a contraction or a rotation.

Lemma 3.2.1 (Schwarz Lemma)**.** *Let $f : \mathbb{D} \to \overline{\mathbb{D}}$ be a holomorphic function so that $f(0) = 0$. Then we will have that $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Moreover if $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$, then $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$.*

Proof. Define

$$g(z) := \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

Note we have that

$$\lim_{z\to 0} g(z) = \lim_{z \to 0} \frac{f(z)}{z} = \frac{f(z) - 0}{z - 0} = f'(0) = g(0)$$

so that g is continuous at 0. In particular since g is holomorphic elsewhere, this means that g is holomorphic at 0, and moreover holomorphic on all of $\mathbb{D}$.

Now, let $D_r = \overline{D}(0; r)$ for $0 < r < 1$. By the Maximum Principle, we know that the maximum of $|g|$ inside D_r must occur on the boundary. This means there is some $z \in \partial D_r$ so that for all $z \in D_r$ we have

$$|g(z)| \leq |g(z_r)| = \frac{|f(z_r)|}{|z_r|} = \frac{|f(z_r)|}{r} \leq \frac{1}{r}$$

If we let $r \to 1$, it then follows that $|g(z)| \leq 1$ for all $z \in \mathbb{D}$, so $|f(z)| \leq |z|$.

Lastly if we have that $|f(z)| = |z|$, or that $|f'(0)| = 1$, then it follows that $|g(z)| = 1$ for some $z \in \mathbb{D}$. Since $|g(z)| \leq 1$, then g attains its maximum in its interior. By the Maximum Principle, this can only occur if g is constant, so that $f(z) = az$ for some $a \in \mathbb{C}$. It follows trivially that $|a| = 1$. □

We shall also use need following set of biholomorphic functions on the unit disk.

Definition 3.2.2. *We define an automorphism of the unit disk $g_c : \mathbb{D} \to \mathbb{D}$ as*

$$g_c(z) := \frac{z - c}{\bar{c}z - 1}$$

for any $c \in \mathbb{D}$.

Lemma 3.2.3. *For any $c, z \in \mathbb{D}$ we have $g_c(g_c(z)) = z$, and is therefore a bijection.*

Proof. The first part of the proof is a routine calculation

$$\begin{aligned}
g_c(g_c(z)) &= \frac{g_c(z) - c}{\bar{c}g_c(z) - 1} \\
&= \frac{\dfrac{z-c}{\bar{c}z-1} - c}{\bar{c}\dfrac{z-c}{\bar{c}z-1} - 1} \\
&= \frac{z - c - c(\bar{c}z - 1)}{\bar{c}(z-c) - (\bar{c}z - 1)} \\
&= \frac{z - c - |c|^2 z + c}{\bar{c}z - |c|^2 - \bar{c}z + 1} \\
&= \frac{z - |c|^2 z}{1 - |c|^2} \\
&= z
\end{aligned}$$

From this we can see that g_c is its own inverse, meaning g_c is bijective. □

One consequence of Schwarz Lemma is the following lemma. This lemma tells us that a particular function is in fact a contraction, and we shall need this fact later on.

Lemma 3.2.4. *Let j be the squaring function (i.e. $j(z) = z^2$), and take any $c \in \mathbb{D}$. Then the function $\psi_c := g_{c^2} \circ j \circ g_c$ is a proper contraction in the sense that $\psi_c(0) = 0$ and*

$$|\psi_c(z)| < |z|$$

for all $z \in \mathbb{D} \setminus \{0\}$.

Proof. Firstly, we can easily check that indeed $\psi_c(0) = 0$. By Schwarz Lemma, we then know that $|\psi_c(z)| \leq |z|$. Suppose that $|\psi_c(z)| = |z|$ for some $z \in \mathbb{D} \setminus \{0\}$. Then by Schwarz again we would have that $\psi_c(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$. However, we can see that this cannot be the case as j is not bijective, and hence ψ_c cannot be bijective either. Therefore we must have that $|\psi_c(z)| < |z|$ for all non-zero $z \in \mathbb{D}$. □

3.3 Holomorphic Injections

We now want to look at biholomorphic functions on a fixed Q-domain to the unit disk. However, instead of directly dealing with biholomorphic functions, we focus our attention on injective holomorphic functions. We define the following family of functions.

Definition 3.3.1. *Fix any Q-domain $G \neq \mathbb{C}$. We define the collection of functions $\mathcal{F}_G$ as*

$$\mathcal{F}_G = \{f : G \to \mathbb{D} \mid f \text{ is holomorphic and injective}, f(0) = 0\}$$

The Riemann Mapping Theorem can then be stated by saying that there is some $f \in \mathcal{F}_G$ so that $f(G) = \mathbb{D}$. We shall prove this in the following section.

Firstly, we note that $\mathcal{F}_G$ is non-empty for any $G \neq \mathbb{C}$.

Lemma 3.3.2. *Let $G \neq \mathbb{C}$ be a Q-domain. Then there is an injective holomorphic function $f : G \to \mathbb{D}$ such that $f(0) = 0$.*

Proof. Take any $a \in \mathbb{C} \setminus G$. It follows that $z - a$ is non-vanishing on G, so there is a holomorphic function $v : G \to \mathbb{C}$ such that $v(z)^2 = z - a$.

Now, suppose that $v(z_0) = \pm v(z_1)$ for some $z_0, z_1 \in G$. Then we must have that

$$z_0 - a = v(z_0)^2 = (\pm v(z_1))^2 = z_1 - a$$

and so $z_0 = z_1$, which implies that v is injective. Moreover it means that if $v(z_0) = -v(z_1)$, then $z_0 = z_1$, so we would have that $v(z_0) = -v(z_0)$, and so $v(z_0) = 0$. But this would mean that $v(z_0)^2 = z_0 - a = 0$, which cannot hold as $a \notin G$ so $v(z_0) \neq 0$. In other words, we have

$$v(G) \cap (-v)(G) = \emptyset$$

Now holomorphic functions are open maps (Theorem 1.1.6), so $(-v)(G)$ will be open (and trivially non-empty). So there is a disk $D = D(c; r)$ contained in $(-v)(G)$. It then also follows that $v(G) \subseteq \mathbb{C} \setminus \bar{D}$. So if we define the function

$$g(z) := \frac{r}{2}\left(\frac{1}{z - c} - \frac{1}{v(0) - c}\right)$$

then g maps $v(G) \subseteq \mathbb{C} \setminus \bar{D}$ injectively into $\mathbb{D}$.

Defining $f := g \circ v$, we see that $f(0) = 0$, and that f maps G injectively into $\mathbb{D}$. $\square$

Our next step is to maximise the size of $f(G)$ for $f \in \mathcal{F}_G$. We do this by maximising the value of $|f(p)|$ for any $p \neq 0$. Importantly, we observe that there is a function $f \in \mathcal{F}_G$ that attains this maximum.

Lemma 3.3.3. *Take any $p \neq 0$ in G. Let $\mu := \sup\{|h(p)| : h \in \mathcal{F}_G\}$. Then there is a $f \in \mathcal{F}_G$ so that $|f(p)| = \mu$.*

Proof. Let f_n be a sequence in $\mathcal{F}_G$ so that $|f_n(p)|$ converges to μ. Since f_n map to $\mathbb{D}$, this sequence is bounded. By Montel (Theorem 2.1.7), this means it has a subsequence which converges compactly to some holomorphic function f. From this we know that $|f(p)| = \lim |f_n(p)| = \mu$. All that remains is to check that $f \in \mathcal{F}_G$.

Firstly we will clearly have that $f(0) = 0$. We also know that each f_n is injective, and that moreover f is non-constant as $f(p) \neq f(0)$. By Hurwitz's Injection Theorem (Theorem 2.2.3) we then have that f is injective and $f(G) \subseteq \mathbb{D}$. Hence $f \in \mathcal{F}_G$. $\square$

As it turns out, such a maximising function will have that $f(G) = \mathbb{D}$. In other words, this maximal function is in fact the biholomorphic mapping we were looking for. This is shown in the following lemma.

Lemma 3.3.4. *Let $f \in \mathcal{F}_G$ so that $|f(p)| = \sup\{|h(p)| : h \in \mathcal{F}_G\}$ for some $p \neq 0$. Then $f(G) = \mathbb{D}$.*

Proof. Let $H = f(G) \subseteq \mathbb{D}$. Note that H will also have the square-root property. Suppose that $H \neq \mathbb{D}$. Then there must be some element in $\mathbb{D}$ not in H. In particular, choose $c \in \mathbb{D}$ so that $c^2 \notin H$.

If we restrict the automorphism g_{c^2} on H, then we can see that as $c^2 \notin H$, g_{c^2} will be non-vanishing. This means it will have a square root v. In particular, since $g_{c^2}(0) = c^2$, we can choose the root which has $v(0) = c$. Moreover we must have that $v(H) \subseteq \mathbb{D}$ as $v^2(H) = g_{c^2}(H) \subseteq \mathbb{D}$.

Now, we define $\kappa := g_c \circ v$ (note this is well defined as $v(H) \subseteq \mathbb{D}$). We claim that κ is an expansion, and we shall show this by considering ψ_c as in Lemma 3.2.4. We then have that

$$\psi_c \circ \kappa = (g_{c^2} \circ j \circ g_c) \circ (g_c \circ v) = g_{c^2} \circ j \circ (g_c \circ g_c) \circ v = g_{c^2} \circ j \circ v = g_{c^2} \circ g_{c_2} = id_{\mathbb{D}}$$

Since ψ_c is a contraction, this means that for any non-zero z we have that $|z| = |\psi_c(\kappa(z))| < |\kappa(z)|$. This in fact also shows us that κ is injective. This is because if $\kappa(z_1) = \kappa(z_2)$, then

$$z_1 = \psi_c(\kappa(z_1)) = \psi_c(\kappa(z_2)) = z_2$$

Now, let $h := \kappa \circ f$ (again, well defined as $f(G) = H$). We firstly know that

$$h(0) = \kappa(f(0)) = \kappa(0) = g_c(v(0)) = g_c(c) = 0$$

Moreover since both κ and f are injective, then so is h. Lastly, since $h = \kappa \circ f = g_c \circ v \circ f$, and the image of g_c is in $\mathbb{D}$, then $h(G) \subseteq \mathbb{D}$. Therefore $h \in \mathcal{F}_G$.

However, we have that

$$|h(p)| = |\kappa(f(p))| > |f(p)|$$

This contradicts the maximality property of f. Hence our assumption was incorrect, and therefore $f(G) = \mathbb{D}$. □

As an interesting aside, when we found this maximal function f, we maximised the value of $|f(p)|$ in $\mathcal{F}_G$. However this is not the only way we could have found a maximal function with the properties above. An alternative way would be to maximise on $|f'(0)|$ instead, and we would still obtain the same result (we chose $|f(p)|$ as it had to a slightly more concise proof).

We now have enough to prove the Riemann Mapping Theorem.

Theorem 3.3.5 (Riemann Mapping Theorem)**.** *Let $G \neq \mathbb{C}$ be a simply connected region. Then there is a biholomorphic function $f : G \to \mathbb{D}$.*

Proof. We can assume without loss of generality that $0 \in G$ (if not we can simply take a shifted version of G which does include 0). This then means that by Lemma 3.1.5, G is a Q-domain.

Now, from the previous section, we have seen that there is a holomorphic function $f : G \to \mathbb{D}$ so that f is injective and $f(G) = \mathbb{D}$. Since this is the case, then f is surjective onto $\mathbb{D}$, and therefore bijective. So by Lemma 3.1.3, f is a biholomorphic function. □

We have shown the existence of such a function. It turns out that such a function is almost unique. It may vary by choosing which point is mapped to origin, as well as a rotation.

Theorem 3.3.6 (Uniqueness)**.** *Let $G \neq \mathbb{C}$ be a simply connected region and take any $a, b \in G$ with $a \neq b$. Then there is exactly one biholomorphic map $h : G \to \mathbb{D}$ so that $h(a) = 0$ and $\arg(h(b)) = 0$.*

Proof. Assume without loss of generality that $a = 0$ so that G is a Q-domain. We shall consider $\mathcal{F}_G$ with our fixed point being $p = b \neq 0$.

We shall first show existence of such a function. We can find $f \in \mathcal{F}_G$ so that $|f(b)|$ is maximal, and we know that f will map G' biholomorphically onto $\mathbb{D}$. Define

$$h(z) = \frac{|f(b)|}{f(b)} f(z)$$

so that $\arg(h(b)) = 0$. Hence h has this property.

Now suppose that h_1 also has this property. Then define $g := h \circ h_1^{-1}$. We can see that $g : \mathbb{D} \to \mathbb{D}$ is a holomorphic function with $g(0) = 0$. If we let $c = h_1(b)$, then we can see that $|g(c)| = |h(b)| \geq |h_1(b)| = |c|$. But by Schwarz Lemma we must have that $|g(c)| \leq |c|$. Therefore $|g(c)| = |c|$, so we get by Schwarz again that $g(z) = kz$ for some $k \in \mathbb{C}$ with $|k| = 1$.

Now, we know that both $h(b)$ and $h_1(b)$ are both real and positive. Now since

$$h(b) = g(c) = kc = kh_1(b)$$

then it follows that $k = 1$, and hence g is the identity function. So $h = h_1$, and we have uniqueness. $\square$

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