Amenability of the Central Fourier Algebra

Aleksa Vujičić

University of Waterloo

30 August 2023

Overview

- Preliminaries
- Brief history of problem
- Introduce example
- Proof in two parts
- Current and future work

Preliminaries (groups)

- Our starting point is a locally compact group (LCG) G.
- We will need two important definitions relating to LCGs: *amenable* and *virtually abelian*.
- I will not define amenability here, but it encompasses a lot of nice classes of groups (compact, abelian).
- We say G is *virtually abelian* if it has a closed abelian subgroup of finite index.
- If G is compact, being virtually abelian is equivalent to finding open abelian subgroup.

Preliminaries (groups)

- Groups also have important Banach algebras:
- The *group algebra* $L^1(G)$:
  - Multiplication is given by convolution

$$f * g(x) = \int_G f(s)g(s^{-1}x)ds$$

  - The norm is given by the usual L^1 norm $\|f\|_1 = \int_G |f(s)|ds$.
- The *Fourier algebra* $A(G)$:
  - Defined by

$$A(G) = \overline{\text{span}}\left\{x \mapsto \int_G f(x^{-1}s)g(s)ds : f, g \in L^2(G)\right\} \subseteq C_0(G)$$

  - Multiplication is given by pointwise multiplication of functions.
  - I will skip defining the norm here.

Preliminaries (abelian groups)

- When G is abelian, we can define a (Pontryagin) dual group $\hat{G}$, which contains all continuous homomorphisms of G onto the unit circle $\mathbb{T}$.
- With pointwise multiplication, this is also an abelian LCG.
- The *Fourier transform* is an isomorphism from $L^1(G)$ to $A(\hat{G})$ defined by:

$$\hat{f}(\varphi) = \int_G f(x)\varphi(x)dx$$

- We have that $\hat{\hat{G}} \simeq G$, and under this equivalence we have $\hat{\hat{f}} = f$.

Preliminaries (bads)

- Given a Banach algebra $\mathcal{A}$, we can define bi-module actions on $\mathcal{A}\hat{\otimes}\mathcal{A}$.
- Specifically for $a, b, c \in \mathcal{A}$:
  - $a \cdot (b \otimes c) = (ab) \otimes c$
  - $(b \otimes c) \cdot a = b \otimes (ca)$
- We also define the diagonal operator $m : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ by $m(a \otimes b) = ab$.
- A *bounded approximate diagonal* (BAD) on $\mathcal{A}$ is a net $(\mu_\gamma)_\gamma \in \mathcal{A}\hat{\otimes}\mathcal{A}$ such that for every $a \in \mathcal{A}$:
  - $a \cdot \mu_\gamma - \mu_\gamma \cdot a \to 0$
  - $am(\mu_\gamma) = a$
- We say $\mathcal{A}$ is amenable if it admits a BAD.

Brief History

Theorem (Johnson 72)

Let G be a LCG. Then G is amenable if and only if $L^1(G)$ is amenable

- The case for $A(G)$ is similar but slightly different.

Theorem (Leptin 68)

Let G be a LCG. Then G is amenable if and only if $A(G)$ admits a bounded approximate identity.

- There is also a similar notion in terms of operator amenability.

Theorem (Ruan 95)

Let G be a LCG. Then G is amenable if and only if $A(G)$ is operator amenable.

- What happens if $A(G)$ is amenable instead?

Brief History

- Henceforth we assume G is compact.
- We have the following for $A(G)$:

Theorem (Forrest, Runde 2004)

G is virtually abelian if and only if $A(G)$ is amenable.

- Conjecture first posed by Azimifard-Samei-Spronk in 2009.

Conjecture

G is virtually abelian if and only if $ZL^1(G)$ is amenable.

- Forward direction was proved by Alaghmandan-Crann in 2017.

Brief history

- Related conjecture posed by Alaghmandan-Spronk in 2017.
- We define a subalgebra of $A(G)$ called the **central Fourier algebra**:

$$\begin{aligned} ZA(G) &= \{u \in A(G) : u(xyx^{-1}) = u(y), \forall x, y \in G\} \\ &= A(G) \cap ZL^1(G) \end{aligned}$$

Conjecture

G is virtually abelian if and only if $ZA(G)$ is amenable.

- Forward direction proved in same paper.
- Reverse is known to be true in nice cases:
  - Connected groups
  - (Infinite) product of finite non-abelian groups.

An example towards an idea

- Consider the p-adic group $G = \mathbb{O}_p \rtimes \mathbb{O}_p^* = \begin{pmatrix} \mathbb{O}_p^* & \mathbb{O}_p \\ 0 & 1 \end{pmatrix}$.
- This is a compact, totally disconnected, non virtually abelian group.
- Consider $ZA(G)$:

$$\begin{aligned} ZA(G) &= \{u \in A(G) : u((x,a)(y,b)(x,a)^{-1}) = u((y,b)), \\ &\qquad \forall (x,a),(y,b) \in \mathbb{O}_p \rtimes \mathbb{O}_p^*\} \\ &= \{u \in A(G) : u(((1-b)x + ay, b)) = u((y,b)), \\ &\qquad \forall (x,a),(y,b) \in \mathbb{O}_p \rtimes \mathbb{O}_p^*\} \end{aligned}$$

- So we are interested in all functions which are constant on subsets given by the above equation.
- What if we restrict our attention to the case where $b = 1$?

A new algebra

- Consider the subgroup $H = \mathbb{O}_p \rtimes \{1\}$.
- If $u \in ZA(G)$, then $u\big|_H$ must satisfy $u((ay, 1)) = u((y, 1))$ for all $a \in \mathbb{O}_p^*$ and $y \in \mathbb{O}_p$.
- So we get a quotient algebra of $ZA(G)$ defined as follows:

$$ZA(G)\big|_H = \{u \in A(G) : u(ay, 1) = u(y, 1), \forall x \in \mathbb{O}_p, a \in \mathbb{O}_p^*\}$$

- We see a natural isomorphism to the algebra:

$$Z_{\mathbb{O}_p^*}A(\mathbb{O}_p) = \{u \in A(\mathbb{O}_p) : u(ax) = u(x), \forall x \in \mathbb{O}_p, a \in \mathbb{O}_p^*\}$$

- Notice these are the functions in $ZA(\mathbb{O}_p)$ which are "invariant" under the action of $\mathbb{O}_p^*$.
- This will be our focal algebra.

A shadow diagonal

- The orbits of $\mathbb{O}_p^* \curvearrowright \mathbb{O}_p$ are $B_m := p^m\mathbb{O}_p \setminus p^{m+1}\mathbb{O}_p = p^m\mathbb{O}_p^*$ and $\{0\}$.
- Each B_m is open, and has a minimal idempotent $1_{B_m} = 1_{p^m\mathbb{O}_p} - 1_{p^{m+1}\mathbb{O}_p} \in Z_{\mathbb{O}_p^*}A(\mathbb{O}_p)$.
- We can construct a "shadow diagonal" on the open orbits:

$$e_n = \sum_{m=0}^{n-1} 1_{B_m} \otimes 1_{B_m} \in Z_{\mathbb{O}_p^*}A(\mathbb{O}_p)\hat{\otimes} Z_{\mathbb{O}_p^*}A(\mathbb{O}_p)$$

- We can also define the f.d. ideal $A_n = \mathrm{span}\{1_{B_m} : 0 \leq m < n\}$, with identity

$$\sum_{m=0}^{n-1} 1_{B_m} = 1_{\mathbb{O}_p} - 1_{p^n\mathbb{O}_p}$$

- Identity has norm at most 2.

Boundedness along this shadow diagonal

- Suppose $Z_{\mathbb{O}_p^*}A(\mathbb{O}_p)$ is amenable, so it has a BAD $(\mu_\gamma)_\gamma$.
- Project $(\mu_\gamma)_\gamma$ onto $A_n\hat{\otimes}A_n$, and let η_n be a cluster point.
- We have η_n is bounded since identity of A has norm at most 2.
- Can compute coefficients to show that $\eta_n = e_n$.
- So e_n is bounded.

Unboundedness of this shadow diagonal

- We can also compute e_n explicitly using the Fourier transform.
- The dual group of $\mathbb{O}_p$ is $\mathbb{T}_{p^\infty} = \{x \in S^1 : \exists k \in \mathbb{N}, x^{p^k} = 1\}$.
- Recall we defined e_n as

$$e_n = \sum_{m=0}^{n-1} 1_{B_m} \otimes 1_{B_m} = \sum_{m=0}^{n-1} (1_{p^m\mathbb{O}_p} - 1_{p^{m+1}\mathbb{O}_p}) \otimes (1_{p^m\mathbb{O}_p} - 1_{p^{m+1}\mathbb{O}_p})$$

- Let $H_n = \{x \in \mathbb{T}_{p^\infty} : x^{p^n} = 1\}$.
- The Fourier transform of e_n is

$$d_n = \sum_{m=0}^{n-1} (\frac{1}{p^m} 1_{H_m} - \frac{1}{p^{m+1}} 1_{H_{m+1}}) \otimes (\frac{1}{p^m} 1_{H_m} - \frac{1}{p^{m+1}} 1_{H_{m+1}})$$

- Since $\mathbb{T}_{p^\infty}$ is discrete, it is possible to compute norm directly.

Unboundedness of this shadow diagonal

- Given $x \in \mathbb{T}_{p^\infty}$, we say it has **order** k if $x^{p^k} = 1$ (and k is minimal).
- With some non-trivial computation, one can show

$$|d_n(x,x)| \geq p^{-2k} - p^{-2n}$$

  where $x \in \mathbb{T}_{p^\infty}$ has order $k \leq n$.
- Use this to find lower bound

$$\|d_n\| \geq (1 - p^{-1})^2 \left(n - \frac{p^2 - p^{-2n}}{p^2 - 1} \right)$$

- So d_n (and hence e_n) is unbounded.

Summary & current progress

- All BAD's of $Z_{\mathbb{O}_p^*}A(\mathbb{O}_p)$ must 'asymptotically approach' e_n.
- But e_n is unbounded, so no B.A.D.'s exist.
- Thus $ZA(\mathbb{O}_p \rtimes \mathbb{O}_p^*)$ is not amenable, affirming the conjecture.
- This works if we replace $\mathbb{O}_p$ with another compact discrete valuation ring.
  - The other common example of this is the formal power series ring $F[[X]]$ over a finite field F.

Future work

- Hope to use this technique on a broader class of groups.
- What kind of group actions (along with their orbit structure) allow this technique to work?
- It is likely necessary that the group action has a lot of open orbits.
- What about other subgroups of $GL_n(\mathbb{O}_p)$ or $GL_n(R)$ (where R is a CDVR)?

End

Thank you for listening!